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THE APPLICATION OF MATHEMATICS TO THE SOCIAL SCIENCES*

BY IRVING FISHER

The invitation to deliver this lecture was accepted chiefly because of my veneration for J. Willard Gibbs, whose pupil I was forty years ago.

It was by accident rather than by design that my own life work was diverted from mathematical physics and mathematics, which it was my privilege to study under Gibbs, and was turned, instead, toward the application of mathematics to the social sciences.

While I have lost touch with the subject of which J. Willard Gibbs was a master, the debt which I owe to him and the help which my studies with him have afforded me even in the field of the social sciences can never be forgotten.

I hope, therefore, that it may not be amiss to precede what I have to say on Mathematics in the Social Sciences by a reminiscent statement of my personal impressions of Gibbs himself. J. Willard Gibbs towered, head and shoulders, above any other intellect with which I have come in contact. I had a keen realization of his greatness even in those formative years in Yale College and the Yale Graduate School. But this keen realization has grown even keener as the years have swept by, not only because of the increased evidence of the fundamental value of Gibbs' work in his own chosen field but also because in my own consciousness, after so many details have dropped from memory, there persists all the more clearly the strong impression which Gibbs' personality and teaching made upon me.

In saying this I do not think I can be accused of undue enthusiasm simply from the loyalty of a pupil to his teacher, especially in view of the statements of Lord Kelvin and others, which virtually rank Gibbs as the Sir Isaac Newton of America. Lord Kelvin said when visiting at Yale, a few years ago, that

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* The seventh Josiah Willard Gibbs Lecture, read at Des Moines, December 31, 1929, before a joint session of the American Mathematical Society and the American Association for the Advancement of Science.

"by the year 2000 Yale would be best known to the world for having produced J. Willard Gibbs."

One of the most striking characterizations of Gibbs was recently made by Dr. John Johnston, now with the United States Steel Corporation, then Professor of Chemistry at Yale, in his address on Gibbs delivered at Yale University two years ago. He stated that no result of Gibbs' work had yet been overthrown, and that, in this respect, Gibbs seems to stand unique and supreme among the great scientists.

The English physical chemist, Professor F. G. Donnan, has said that "Gibbs ranks with men like Newton, Lagrange and Hamilton, who by the sheer force and power of their minds have produced those generalized statements of scientific law which mark epochs in the advance of exact knowledge . . . . . The work and inspiration of Gibbs have thus produced not only a great science but also an equally great practice. There is, today, no great chemical or metallurgical industry that does not depend, for the development and control of a great part of its operations, on an understanding and application of dynamic chemistry and the geometrical theory of heterogeneous equilibria."

Professor Ostwald said, in the preface to his German translation of Gibbs' thermodynamic papers in 1892:

"The importance of the thermodynamic papers of Willard Gibbs can best be indicated by the fact that in them is contained, explicitly or implicitly, a large part of the discoveries which have since been made by various investigators in the domain of chemical and physical equilibrium and which have led to so notable a development in this field . . . . . The contents of this work are today of immediate importance and by no means of merely historical value. For of the almost boundless wealth of results which it contains, or to which it points the way, only a small part has up to the present time 1892 been made fruitful."

Sir Joseph Larmor said:

"This monumental memoir *On the Equilibrium of Heterogeneous Substances* made a clean sweep of the subject and workers in the modern experimental science of physical chemistry have returned to it again and again to find their

empirical principles forecasted in the light of pure theory and to derive fresh inspiration for new departures."

We think no less of Gibbs' greatness because he himself showed so little consciousness of it. He must have realized the fundamental character of his work. But his pupils remarked his profound modesty and often commented on it. His chief delight was in truth seeking for its own sake, and he was so intent on this search that he had no time even to think of emphasizing the originality or value of his own additions to the great vista of truth over which his mind swept. Doubtless he often did not know or greatly care where the work of others ceased and his own began. He did not always wade through the literature which preceded his own scientific papers. I remember hearing him say that when he wanted to verify another man's results he usually found it easier to work them out for himself than to follow the other man's own course of reasoning. This was said without any affectation, but simply in a jocular vein, as by one who would escape a difficult task by going his own way. But even though it may be difficult to disentangle what was original in Gibbs' work from what was anticipated by others, surely no competent person doubts today that he founded a new era in physics and chemistry.

Because of his retiring disposition and the theoretical nature of his work, he was, during his lifetime, almost unknown except among a few special students. The majority of students at Yale, in my day, did not know of his existence, much less of his greatness. And there were far fewer people in America who could appreciate what he was doing than there were in Europe.

His work was more promptly recognized in Germany. When I studied in Berlin in 1893, and was asked under whom I had studied in America, I enumerated the mathematicians at Yale. To my mortification not one of the names was known to those Berlin professors, until I mentioned Gibbs, whereupon they were loud in his praises. "*Geebs, Geebs, jawohl, ausgezeichnet*!"

Even today, Gibbs illustrates the rule that a prophet is not without honor save in his own country. As Professor Johnston noted, the fiftieth anniversary of the publication of the first part of Gibbs' great work on the Equilibrium of Heterogeneous Substances was signalized in Holland by the publication of a

Gibbs' number of their chemical journal. This contained contributions from Dutch authorities, as well as from French, German, Canadian, Norwegian and English authorities but none from an American!

It is true, however, that at Yale we have finally established a Gibbs fund for a lecturership to be filled by visiting professors and that a new complete edition of his works has been issued by Longmans, Green and Co. It is also planned to issue two volumes of commentations on Gibbs' work to make its chief results more accessible to the general scientist. I am proud to have played a part in these undertakings. May I take this opportunity to say that I am also proud to have been included among the J. Willard Gibbs lecturers? And may I congratulate the American Mathematical Society on being the organization to found this lecturership, although Gibbs was professedly not so much a mathematician as a physicist. The only other Gibbs lecturership seems to be that at the Mellon Institute in Pittsburgh. The Chicago Section of the American Chemical Society awards a "J. Willard Gibbs medal" annually, the recipient of which makes an address.

Presumably Gibbs' greatest contribution to science was his application of the laws of thermodynamics to chemistry. He made this almost a deductive science. Professor Bumstead said of his work: "To an unusual extent, among the sciences which appeal to experiment, it can be, and has been, cast in a deductive form. Sir Isaac Newton said that 'it is the glory of geometry that from a few principles it is able to produce so many things.' Thermodynamics shares in this kind of glory; it has only two fundamental principles, of which the first is a statement of the conservation of energy as applied to heat, and the second states the fact (so deeply founded in general experience that it seems almost axiomatic) that heat will not, of itself, flow from a body at a lower temperature to one of a higher temperature. From these two simple principles, by an almost euclidean method, a surprising number of facts and relations between work and heat, and various properties of bodies were deduced about the middle of the last century."

J. Willard Gibbs was certainly a master at producing many deductions from a few general principles. And it was just because of the generality of the principles from which he

always insisted on starting that he succeeded in reaching such a wealth of conclusions.

In fact, it has always seemed to me that Gibbs' chief intellectual characteristic consisted in his tendency to make his reasoning as general as possible, to get the maximum of results from the minimum of hypotheses. I shall never forget, and have often quoted, an aphorism used by Gibbs, whether or not original with him, to the effect that "the whole is simpler than its parts." For instance, when he had a problem involving coordinates he preferred to employ an *indeterminate* origin, maintaining that his results were thereby rendered simpler and easier than if he took the origin at some apparently more convenient but special point in relation to the crystal or other conformation which he was discussing. When the origin is indeterminate it automatically effaces itself from all the general relations deduced.

Many, if not most, other investigators instinctively seek to solve special cases before attempting to solve the general case. Sometimes they pay a big penalty in needless experimentation. I remember Professor Bumstead, my fellow student at Yale, recounting with relish a conversation that Gibbs was reputed to have had with a youthful investigator who had made a laborious experimental study of certain relationships and who was, with pardonable pride, telling Gibbs of his conclusion. After listening attentively Professor Gibbs quite naturally and unaffectedly closed his eyes, thought a moment, and said, "Yes, that would be true", seeing at once that the special result which this young investigator had reached was a necessary corollary of Gibbs' own more general results. For him, it required no experimental verification. The young man's work had, from Gibb's viewpoint, been almost as much wasted as if it had been spent in a laborious set of measurements of right angle triangles on the basis of which measurements he should announce as a new and experimental discovery that the square of the hypotenuse is equal to the sum of the squares of the other two sides. It is worth noting that, though Gibbs did his work in the Sloane Physics Laboratory, he never, as far as I know at least, performed a single experiment. His life work, stupendous as it was, and based as it was on concrete fact, consisted ex-

clusively in new deductions from old results, the full significance of which no one else had been able to derive.

In his effort to represent physical relations by geometrical models and to portray the theory of electricity and magnetism by geometrical methods, Gibbs encountered the need of a new vector analysis to replace the awkward analysis by Cartesian coordinates, requiring, as that does, three times as many equations to write and manipulate as does vector analysis, not to say diverting attention from the lines and surfaces actually involved to their projections on three arbitrary axes.

To me the most interesting course I had with Gibbs was on vector analysis. He believed he had simplified the Hamilton system of quaternions, getting his cue from Grassmann's *Ausdehnungslehre*. But he was so conscious of his obligations to Grassmann that he was reluctant to publish his own system, apparently doubting whether it possessed enough originality to warrant publication. He therefore had privately printed a syllabus of his system and this reprint was used by us in his class as a text. Only after many years did Professor E. B. Wilson construct a more elaborate text book embodying Gibbs' principles of vector analysis.

It is a curious fact that, while Gibbs' work in thermodynamics was appreciated in Germany, his work in vector analysis was not. I remember the comment of Professor Schwartz at Berlin, when I undertook to defend Gibbs' vector analysis: "*Es ist zu willkürlich.*" The Germans felt in honor bound to restrict pure mathematics to mere elaboration of the proposition that one and one make two. Starting with this proposition, by successive additions or subtractions of unity, we may, of course, by going forward, obtain all the positive integers; this is addition. Reversing, we obtain zero and the negative integers; this is subtraction. Then, by applying repeated addition or multiplication and repeated subtraction or division, repeated multiplication or involution, and repeated division or evolution, we arrive at fractions, surds, imaginary quantities and finally the *complex variable* $x+yi$. But beyond this, by such processes no more general form of magnitude can possibly be derived; for, if we operate on a complex variable by addition, subtraction, multiplication, division, involution or evolution, under the recognized rules of algebra, we obtain simply other

complex variables and nothing else whatsoever. Only by changing, or as the German critics would say, by violating, the fundamental rules of algebra faithfully followed in the above processes, such as the rule that $a \times b$ is equal to $b \times a$, is it possible to enter into any other realm of mathematics than that of the complex variable.

When I reported these criticisms to Gibbs, his comment was that all depends on what your object is in making those sacrosanct rules for operating upon symbols. If the object is to interpret physical phenomena and if we find we can do better by having a rule that $a \times b$ is equal not to $b \times a$ but to minus $b \times a$, as in the multiplication of two vectors, then, he said, the criticisms of the Germans are beside the point.

The fact is that Gibbs, though a great mathematician, was not primarily interested in mathematics as such. His interest lay in its applications to reality—in the substance rather than the form. All his contributions to pure mathematics were sought and found not as mere proliferations of formal and abstract logic but as by-products of his work in interpreting the facts of the physical universe.

The far reaching effects of Gibbs' work apply not only to inorganic physics and chemistry but also to the organic world. One of the most elaborate reviews of Gibbs and his relation to modern science is by Lt. Col. Fielding H. Garrison, M.D., Assistant to the Librarian of the Army Medical Library, Washington, D. C., in which he shows, among other things, the application of Gibbs' work to the equilibrium of heterogeneous substances in general physiology.

Despite Gibbs' retiring disposition and his disinclination for general society he was most cordial in his personal contact with colleagues and students and never seemed to lack time to give to anyone who chose to discuss the subject in which he was so deeply interested. He made on all a deep impression of kindliness. I well remember the remark of Percey Smith, now Professor of Mathematics at Yale, who was a fellow student of Gibbs with me, as we walked out of one of Gibbs' lectures, "What a *gentle* man he is!"

He enjoyed a joke, often laughed and excited laughter. He took pleasure in applying his mathematics in simple ways. One of his minor but fascinating papers before the Yale Mathemati-

cal Club was on the *Paces of a horse*, the writing of which was doubtless suggested by watching a horse which he had just purchased. Probably no one else ever put a horse through his paces as scientifically or amusingly as Gibbs did in that paper.

Gibbs himself never contributed to the social sciences. Apparently I am the only one of his pupils who, after first doing some teaching in mathematics and physics, became professedly an economist, although Professor E. B. Wilson, Gibbs' chief interpreter as to mathematics, has taken a lively interest in many lines of social science and statistics and was this year President of the American Statistical Association.

After several years' graduate study partly in mathematics under Gibbs, and partly in economics under Sumner, the time came for me to write my doctor's thesis and I selected as my subject *Mathematical Investigations in the Theory of Value and Prices*. Professor Gibbs showed a lively interest in this youthful work, and was especially interested in the fact that I had used geometric constructions and methods, including his own vector notation.

The late Professor Allyn Young of Harvard also made occasional use of vectors in his economic work. Another economic student and writer, a brilliant young Norwegian, Professor Ragnar Frisch, has latterly used the vector notation and says he could scarcely think without it. Professor Frisch will this year be visiting Professor at Yale from the University of Oslo.

It is one of the handicaps of mathematics in the social sciences that there are so few who are trained in both lines for such study, and this particularly applies to any applications of Professor Gibbs' vector analysis. If vector analysis should become more widely understood and used by students in the social sciences, doubtless it would be more generally utilized, at least as a vehicle for thought.

Occasionally, and increasingly, the ideas and notations of the differential and integral calculus are applied by mathematical economists and statisticians. But, of course, most of the mathematics employed in the social sciences consists of simple algebra. There is a saying which, by the way, was quoted by Gibbs in his address on Multiple Algebra, that "the human mind has never invented a labor-saving machine equal to algebra."

There are several fairly distinct branches of social science to

which mathematics has been, or may be, applied. The chief of these may be distinguished as (1) pure economics, (2) the "smoothing" of statistical series, (3) correlation, and (4) probabilities, all of which overlap to some extent.

My own chief interest in social science, from a mathematical point of view, has been in the first of these four groups, pure theory.

When I began my work in this field in 1891, mathematics in economic theory was looked at askance, despite the fact that many years before, as early as 1838, Cournot had written his brilliant *Researches into the Mathematical Principles of the Theory of Wealth*. This book later greatly stimulated Professor Edgeworth of Oxford and Professor Marshall of Cambridge, and today is ranked among the economic classics. The same may be said of Jevons' *Theory of Political Economy*, published in 1871. But in 1891, when my own economic studies began, even the work of Cournot was almost unknown to economists, and that of Jevons was little used. If one will turn the pages of the main economic literature of 1891 and earlier, he will find practically no formulas and no diagrams. But Walras and Pareto in Switzerland and Pantaleoni and Baroni in Italy, Edgeworth and Marshall in England, Westergaard and Wicksell in Scandinavia, and a few other students in other countries were using and defending the new method.

When, at the request of Professor Edgeworth, I read a slightly mathematical paper on the *Mechanics of bimetalism* before the Economic Section of the British Association for the Advancement of Science at Oxford in September, 1893, I well remember how, in the discussion of that and other mathematical papers, Professor Edgeworth was, as he expressed it, "damped" by the unfriendly criticism of these new methods by Professor Sidgwick and others.

But little by little, the usefulness of mathematics has come to be appreciated. Besides the older writers already mentioned and Auspitz and Lieben, whose work on price determination of 1889 was one of my first inspirations, there have gradually come into this field many younger writers, among whom may be mentioned Professor Henry L. Moore of Columbia University, Professor J. H. Rogers of the University of Missouri, Professor C. F. Roos of Cornell University, Professor Griffith C.

Evans of Rice Institute, Professor Henry Schultz of the University of Chicago, Professor Harold Hotelling of Stanford University, W. A. Shewhart of the Bell Telephone Laboratories, Professors J. Maynard Keynes, A. C. Pigou, and Arthur L. Bowley of England, Professors Albert Aftalion and Jacques Rueff of France, Professor L. von Bortkiewicz and Dr. Otto Kühne of Germany, Professor Wl. Zawadski of Poland, Professor E. Sluts Ky of Russia, Professor Gustav Cassel of Sweden, Professor Ragnar Frisch of Norway, Dr. Willem L. Valk of Holland, Professors Corrado Gini and Luigi Amoroso of Italy.

And, besides the fact of such accessions to the ranks of the small band of professed mathematical economists, is the even more significant fact that economists in general have not only ceased decrying mathematics but are, in many cases, making some slight use of it themselves.

The late Professor Marshall of Cambridge University was one of the first to perceive what was happening. He said:

"A great change in the manner of thought has been brought about during the present generation by the general adoption of semi-mathematical language for expressing the relation between small increments of a commodity on the one hand, and on the other hand small increments in the aggregate price that will be paid for it: and by formally describing these small increments of price as measuring corresponding small increments of pleasure. The former, and by far the more important, step was taken by Cournot (*Recherches sur les Principes Mathématiques de la Théorie des Richesses*, 1838); the latter by Dupuit (*De la mésure d'utilité des travaux publics*, in the Annales des Ponts et Chaussées, 1844), and by Gossen (*Entwickelung der Gesetze des menschlichen Verkehrs*, 1854). But their work was forgotten; part of it was done over again, developed and published almost simultaneously by Jevons and by Carl Menger in 1871, and by Walras a little later. Jevons almost at once arrested public attention by his brilliant lucidity and interesting style. . . . .

"A training in mathematics is helpful by giving command over a marvellously terse and exact language for expressing clearly some general relations and some short processes of economic reasoning; which can indeed be expressed in ordinary language, but not with equal sharpness of outline. And, what is

of far greater importance, experience in handling physical problems by mathematical methods gives a grasp, that cannot be obtained equally well in any other way, of the mutual interaction of economic changes. The direct application of mathematical reasoning to the discovery of economic truths has recently rendered great services in the hands of master mathematicians to the study of statistical averages and probabilities and in measuring the degree of consilience between correlated statistical tables."

Mathematics serves economic theory in supplying such fundamental concepts based on the differential calculus and also through the process of differentiation solves problems of maxima and minima, as in the simple process of determining formally what is the price that the traffic will bear in order to make profits a maximum.

The chief realm of economic theory to which mathematical analysis of this formal kind applies is that of supply and demand, the determination of prices, the theoretical effect of taxes or tariffs on prices. The results cannot always be reduced to figures but are often useful in terms of mere inequalities.

For instance, among the chief theorems shown mathematically by Cournot are the following:

That a tax on a monopolized article will always raise its price, but sometimes by more and sometimes by less than the tax itself.

That a tax on an article under unlimited competition always raises its price but by an amount less than the tax itself.

That a tax proportional to the net income of a producer will not affect the price of his product.

That fixed charges among costs of production do not affect price nor do taxes on fixed charges.

That opening up free trade in a competitive article between two previously independent markets may decrease the total product.

Among the most surprising paradoxes discovered by the mathematical method is one shown by Edgeworth, that if a monopolist sells two articles, say first and third class railway tickets for which the demand is correlated, it may be possible to tax the third class tickets, at a fixed amount each, with the

result that the monopolist not only pays the tax but lowers the prices of both kinds of tickets.

Familiarity with mathematics will save many confusions of thought. For instance, it is just as important in economics to distinguish between a *high* price and a *rising* price as it is in physics to distinguish between velocity and acceleration. *Rate of price change* has important effects, both theoretically and in practice, on the rate of interest and on the volume of business.

Theoretically the rate of interest ought to be higher during a period of rising prices, or depreciation of the dollar, by an amount equal to the rate of depreciation and it ought to be lower during appreciation.

Practically, however, the rate of interest is slow of adjustment and what is more important, inadequate in adjustment. A mathematical statistical analysis of this slowness and inadequacy helps explain great business upheavals as shown in my new book on *The Theory of Interest*. I may say here, parenthetically, though the case is somewhat different, that the recent crash in the stock market was, in large measure, the price paid for tardiness in raising the rate of interest which should have risen over a year ago but was held down artificially.

Again, mathematics will save the economic student and the student of accounting from the many confusions of double counting, especially in the intricate theory of income.

Another elementary, but important, use of mathematics in economics is in making sure that a problem is determinate by counting and matching the number of independent equations and the number of unknown quantities. A great deal of unnecessary misunderstanding has existed and still exists in economic science as to what determines the rate of interest or other magnitudes in economics. These misunderstandings would not exist if the contestants would take the trouble to express themselves mathematically. If we view the matter mathematically it soon becomes evident that one contestant has seen only one of the determining factors, and the other another, without either of them realizing that both are compatible and needed in a complete economic equilibrium. The concept of economic equilibrium in which many factors act and react on each other is one of the chief elementary contributions

of mathematics to economic theory, and one stressed by Cournot, Walras, Marshall, Pareto and Edgeworth.

Still another use of mathematics is in illustrating geometrically or analytically the fact that a price, or a *marginal utility*, is a function not simply of one but of many variables, the function being purely empirical and incapable of analytical or arithmetical expression. In fact, the economic world is a world of n dimensions.

Thus the marginal utility of bread to John Doe is a function of his quantity not only of bread consumed, but of butter, sugar, and numerous other variables.

I have myself tried to apply these and other mathematical ideas to the formal solution of the problem of prices of commodities, the rate of interest, the relation of capital to income, the purchasing power of money, and what Simon Newcomb, the astronomer-economist, called the equation of societary circulation, now called the equation of exchange (the volume of circulating medium multiplied by its velocity of circulation is equal to the price level multiplied by the volume of trade per unit of time).

Most of these and other applications of mathematics to economic theory consist in short chains of reasoning. Professor Marshall had the impression that only short chains of reasoning could ever be expected in mathematical economics. He said: "It is obvious that there is no room in economics for long trains of deductive reasoning: no economist, not even Ricardo, attempted them. It may indeed appear at first sight that the contrary is suggested by the frequent use of mathematical formulas in economic studies. But on investigation it will be found that this suggestion is illusory, except perhaps when a pure mathematician uses economic hypotheses for the purpose of mathematical diversions; for then his concern is to show the potentialities of mathematical methods on the supposition that material appropriate to their use had been supplied by economic study. He takes no technical responsibility for the material and is often unaware how inadequate the material is to bear the strains of his powerful machinery."

But as time goes on, there appear instances of somewhat longer trains of reasoning.

I may take an example from my own work. I have tried to

show how it is possible to estimate numerically, through suitable mathematical equations, the velocity of the circulation of money. The formula for this was derived through a chain of mathematical reasoning requiring several links and embracing a considerable number of variables of which the chief are the volume of money in circulation, the annual flow of money into and out of banks and the annual cash payments to labor. This problem, by the way, of evaluating the velocity of circulation of money had been pronounced insolvable and, without mathematical analysis, it might well be so considered. Incidentally, the calculations indicate that money in the United States circulates about 25 times a year. In other words, the average dollar stays in the same pocket about two weeks.

To turn to a different example, both Professor Ragnar Frisch and myself, by independent methods, both of them highly mathematical, have shown how, theoretically under certain simple hypotheses, to accomplish the statistical measurement of "marginal utility" or desirability, as a function of one's income. This would determine whether or not it is true that if one man has double the income of another his tax ought to be double, more than double, or less than double in order that he should make the same sacrifice. In other words, it would supply a mathematical criterion by which to judge the justice of a progressive income tax.

I say "would" rather than "does" simply because as yet the statistics available do not seem adequate for any accurate evaluation. But Professor Frisch and I are both hoping to pursue this study further. His and my preliminary results are not inconsistent. My own formula is derived by a chain of mathematical reasoning which results in expressing the ratio of the "marginal utility" of money for a person with a certain income to the "marginal utility" which he would have with a different income in terms of the following elements: those two incomes, the percentages which would be spent on food, rent, etc., under the two respective incomes and the index numbers of prices of food, rent, etc., relatively to another country, serving as a standard of comparison.

Mathematics also helps make clear the "dimensionality" of the magnitudes treated. Thus, the quantity of wheat, its price and its value are three magnitudes as unlike in dimensionality

as time, velocity and distance. The rate of interest has the simplest dimensionality, being, like angular velocity, of dimension t^{-1}.

Mathematics helps us analyze time relationships in general, especially to avoid the old confusion between capital and income, the one relating to an instant of time, the other to a period of time.

Capital-income analysis is a development of the last two score years; but its roots go back generations. Every good treatise on algebra includes the formulas for capitalizing annuities and bonds, the formulas underlying the bond tables used in every broker's office.

While the development of mathematical economics from the theoretical side has been steady and impressive since I was a young man, it has by no means been so rapid as the development of the other three branches to which I have referred.

"Smoothing" statistical data, the fitting of formulas and curves to statistics, has, of course, been one of the aims of statisticians for many generations. In this way we have derived our mortality tables, the basis used by actuaries for calculating life insurance premiums.

I understand that the second J. Willard Gibbs lecture was by Robert Henderson on *Life Insurance as a Social Science and as a Mathematical Problem*. The importance of this branch of our subject does not need to be emphasized in an insurance center like DesMoines.

Actuarial science is, of course, a development of the formulas for capitalization or discount, particularly as applied to annuities, combined with the introduction of the probability element based on mortality statistics. It is essentially an analysis of interest and risk. It could be, and perhaps some day will be, applied to other economic problems besides life insurance, as soon as statistics are adequate for assessing risk numerically in other realms than human mortality. In fact one of the crying needs of economic science is a reliable basis for evaluating risks.

Concurrently with actuarial science has developed a science of mathematics of mortality in relation to population, extending at least back to the days of William Farr, Superintendent of the statistical department of the Registrar General's office of England half a century ago. Today this science has been fur-

ther developed by Knibbs of Australia, Lotka, and Glover in the United States, and others.

Recently, with the development of statistics of industry, the art of curve fitting, by mathematical methods, has grown very rapidly, and examples of it will be found in many current issues of statistical journals. I am, myself, with a collaborator, Max Sasuly, working on a new method of curve fitting aimed to avoid the use of any preconceived formula but letting the statistical data themselves write their own formula, so to speak.

One important phase of curve fitting which links it closely with the study of economic theory is the statistical evaluation of supply and demand curves. Among those who have worked in this field are Professor Henry L. Moore of Columbia University, Professor Henry Schultz of Chicago University, Dr. Mordecai Ezekiel of the United States Department of Agriculture, Professors G. F. Warren, F. A. Pearson and C. F. Roos of Cornell University, and Professor Holbrook Working of Stanford University. Professor Schultz was apparently the first to work out the statistical determination of the effect of the tariff on the price of an imported commodity—sugar.

The third group of mathematical work in the social sciences, the development of correlation, is closely associated with the name of Karl Pearson of the University of London, who is still living. His "correlation coefficient" has become almost a standard procedure in economic statistics as well as in other sciences, including biology, in which he is primarily engaged. Today many, if not most, economists, especially if they work in statistics, are making some use of such correlation coefficients. Through them they have been forced to adopt mathematical aids in spite of old traditions and prejudices.

Professor Warren M. Persons, formerly of Harvard, has worked out correlations with lag showing the interrelations of various economic phenomena in such a way as to serve the purposes of forecasting business conditions. A more elaborate method of correlation has been worked out by Karl Karsten of New Haven, a private statistician, who has made tables of correlation between every pair of available series of economic statistics and has put these together by multiple correlation so as to predict any one of the series from all of the others which are found to serve toward that end. He is now issuing regularly a

forecast of commodity prices, of the volume of business, of stock market trends, and of various other economic factors.

In the study of the so-called "business cycle" and forecasting future fluctuations, mathematical economists and statisticians have made increasing use of what is virtually differentiation or integration. Thus I have emphasized "price-change" as distinct from price, of which it is the differential quotient. Reciprocally Mr. Karsten has applied the idea of "quadrature" to the relations of two statistical series where one is virtually derivable from the other by integration. This means if the curves are cyclical that they are related as are the curves of sines and cosines so that one curve is at zero while the other is at a maximum or minimum.

One development in this field in which I have been especially interested has been the study of the distribution of the lag. This appears to be a skew distribution, but nearly normal if time is plotted on a logarithmic scale.

As already indicated, risk is one of the fundamental elements in the mathematical analysis of actuarial science. It is also, in a different way, through frequency distribution, a fundamental element in correlation analysis. In fact, there have been more or less successful attempts by Karl Pearson to resolve a mortality curve into a sum of several frequency curves and by Arne Fisher to do the reverse, synthesize a set of frequency curves representing mortality from certain causes into the total mortality curve. It may also be pointed out that our second topic, curve fitting and smoothing, whether by least square methods or otherwise, is largely a study in probability.

All this brings us to the fourth chief branch of our subject, the mathematical analysis of probability in general insofar as this relates to social phenomena as embodied in statistics. This has been increasingly studied by many economists, especially the late F. Y. Edgeworth, editor of the Economic Journal. Also mathematical statisticians such as G. Udny Yule, Arthur Bowley, R. A. Fisher, Sir William Beveridge, Truman L. Kelly, A. C. Whitaker, William L. Crum, Thiele and others have done much constructive work in this field.

Professor Vilfredo Pareto tried to work out a formula for statistics of incomes in relation to the number of persons possessing incomes of various sizes, and the Pareto curve has be-

come quite famous. It has been shown, however, particularly by Professor Macaulay of the National Bureau of Economic Research, that the Pareto curve is nothing but the "tail" of a probability curve, although Pareto himself had been loath to admit this. It is true that this particular sort of probability or distribution curve is not "normal" even if the abscissas are plotted on a logarithmic scale. It often happens in statistical series, especially where the frequency distribution lies between zero as one extreme and infinity as the other, that the frequency or probability curve while very skew on the arithmetical scale turns out to be nearly "normal" on the logarithmic scale.

I have, of course, by no means exhausted the list of applications of mathematics to economics, still less to the other social sciences. Many applications have been made which are not easily classified under the four heads I have noted, namely, pure theory, curve fitting, correlation and probabilities.

Of these other and miscellaneous applications, the most important, at least in the field of economics and statistics, seems to be that of index numbers. The theory and practice of index numbers have had a special fascination for many of us because they occupy a tantalizing position on the borderline between a priori rational theory and empirical makeshift and because of the many pitfalls encountered in their study. It is closely related to the subject of probability. In my book on *The Making of Index Numbers*, I have tried to find the best formula for an index number out of several score available formulas.

It is also true, of course, that the last three divisions of our subject, curve fitting, correlation study, and probability, traverse all fields of knowledge. They apply not only to my own branch of the social sciences, economics, but to all others such as sociology, anthropology, and education, as well as to fields outside of social science such as psychology, biology, hygiene and eugenics. In all of these we find a development of mathematical method. Each has its own special concepts, measures, and principles such as the cranial index of anthropology, the intelligence quotient of psychology and education, the Mendelian principle in heredity, and these the mathematician may study in terms of averages, index numbers, correlations, deviations, frequency distributions and otherwise. Just as the

multiplication table is applicable in more than one field of knowledge, so mathematics in general is peculiar to none. Sooner or later every true science tends to become mathematical. The social sciences are simply a little later to be reached than astronomy, physics and chemistry, while the biological sciences are later still.

Scientific method is one and the same, whether employed in such sciences as Gibbs developed, or in others. Mathematical notation is, as Gibbs said, simply a *language*. It is required for the best expression of scientific method when the relations to be expressed become sufficiently involved to require it in preference to ordinary language which is less precise and complete. The outlook is bright for a healthy development of mathematics in the social sciences.

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THE GENERAL THEORY OF FACTORIAL SERIES*

BY TOMLINSON FORT

1. *Introduction.* The subject of this paper is factorial series, a class of series which it seems to me are more interesting than American mathematicians generally realize and which can be made to play a most important part in analysis.

By a factorial series we shall understand a series of the form

$$(1) \qquad c_0 + \sum_{n=1}^{\infty} c_n \frac{(n-1)!}{z(z+1)\cdots(z+n-1)} = \Omega(z),$$

where c_n is a sequence of constants and z a complex variable. The term is sometimes also made to include series in non-inverted factorials or the so-called binomial-coefficient series. I shall use the term only as applying to series (1) and at times to certain generalizations of which I shall speak presently. The terms of (1) are not defined when $z=0, -1, -2, \cdots$. Such points will be understood throughout the paper as excepted in all theorems, and neighborhoods of them in all theorems relating to uniformity.

We notice the simple form of series (1), and comparison with series of negative powers, $c_0+\sum_{n=1}^{\infty} c_n(n-1)!\, z^{-n}$, immediately is suggested. We observe first that the terms of the series of negative powers after the first, but for coefficient, come from successive differentiations of $1/z$. The terms of the factorial series come from taking successive differences of $1/z$. The power series yields itself readily to differentiation and integration and certainly a large part of its usefulness in analysis comes from this fact. The factorial series (1) is not so readily differentiated or integrated. However,

$$\Delta\Omega(z) = \Omega(z+1) - \Omega(z) = -\sum_{n=1}^{\infty} c_n \frac{n!}{z(z+1)\cdots(z+n)},$$

another factorial series of equally simple form, quite a parallel

* An address read before the Society, at the invitation of the program committee, October 26, 1929.

to the differentiation of the power series. The application of the inverse operator, $\sum$, is also immediate. We consequently expect the factorial series to play an important role in that portion of analysis in which these operations are prominent, as for example in the theory of difference equations. We next notice that, whereas the denominators in the power series are successive integral powers of z, in series (1) the successive factors in the denominators, with the possible exception of a finite number at first, increase in absolute value. Hence, we expect series (1) to have certain rapidity of convergence features not possessed by the power series. This is in reality true and it is a fact that many functions asymptotic to divergent series of negative powers or at most obtainable from them by indirect summation methods are representable by convergent factorial series. It is this that gives to factorial series one of their greatest elements of interest. Factorials are asymptotic to exponentials and we expect the series to have properties in common with Dirichlet series.

The form of series (1) is simple. This is what makes it interesting. Generalizations will come to the mind of everyone. Much has been done along this line; sometimes with particular purposes in view and sometimes just to generalize. However, most of the generalizations have sacrificed not only simplicity but some useful property as well. Some generalizations that have been made are as follows. These are not necessarily fundamentally different but seem different in originating idea.

I. The series

$$(1') \qquad c_0 + \sum_{n=1}^{\infty} c_n \frac{\omega^{n-1}(n-1)!}{z(z+\omega)\cdots(z+(n-1)\omega)}$$

has been extensively studied by Nörlund.* He has found it useful for studying the function represented by a factorial series.

II. Replace the sequence $0, 1, 2, \cdots$ by a more general sequence, so that (1) becomes

$$(2) \qquad c_0 + \sum_{n=1}^{\infty} c_n \frac{\lambda_1 \cdots \lambda_n}{z(z+\lambda_1)\cdots(z+\lambda_n)}.$$

* Acta Mathematica, vol. 37 (1914), pp. 327–87.

This series, with various degrees of generality for the sequence λ_n, has been studied by a number of authors among them particularly Landau,* Pincherle† and myself.‡

III. Replace each factor $(z+k)$ of the denominator by a function $S_k(z)$, where

$$S_1(z) = \frac{az+b}{cz+d}, \qquad (ad-cb \neq 0),$$

and

$$S_k(z) = \frac{aS_{k-1}(z)+b}{cS_{k-1}(z)+d}, \qquad (k>1).$$

This has been done by Carmichael.§

IV. Replace the terms of (1) by certain general functions so that the series will include both ordinary Dirichlet series and factorial series. This was done by me a few years ago in the Transactions.¶

V. Replace the terms of (1) by a certain general function having asymptotic properties in common with the terms of (1). This has been done by Carmichael‖ in a series of papers.

VI. Replace the denominators in the terms of (1) by general polynomials in z of successively ascending powers.

This was done at least by Appell** although only secondarily as a generalization of factorial series.

VII. Replace series (1) by series of a general type where the functions represented by the series can be represented by Laplace's integral. This has been done by various authors†† over a long period, although not necessarily aiming directly at

* Münchener Berichte, vol. 36, pp. 151–218.

‡ Palermo Rendiconti, vol. 37, pp. 379–390.

† Transactions of this Society, vol. 31, pp. 233–240. For additional references see Hilb and Szasz: Encyklopädie, Band II_2, Heft 8, pp. 1268–1272.

§ American Journal of Mathematics, vol. 36 (1914), pp. 267.

¶ Transactions of this Society, vol. 23, pp. 26–29.

‖ Transactions of this Society, vol. 17 (1916), p. 207.
American Journal of Mathematics, vol. 39 (1917), p. 385.
American Journal of Mathematics, vol. 40 (1918), p. 113.
American Journal of Mathematics, vol. 42 (1920), p. 76.
This Bulletin, vol. 23 (1917), p. 407.

** Bulletin de la Société Mathématique de France, vols. 48, 51.

†† For example Landau, loc. cit., Nielson, Annales de l'Ecole Normale Supérieure, vol. 19 (1902), pp. 409–453; Pincherle, Bologna Rendiconti, vol. 8 (1904), pp. 5–13; Pincherle, Accademia dei Lincei Rendiconti 1902, pp. 140–141, etc.

factorial series. I mention in particular Widder* who delivered a paper before this Society, February, 1928, on this general topic.

I shall proceed to review briefly the present state of the theory of series (1) and (2).

2. *Convergence.* For the saving of time, the history of convergence theorems will not be traced in detail or completeness.

It is now classic and well known that the region of convergence of (1) is a half-plane bounded by a line perpendicular to the axis of reals. This theorem can be proved in various ways. The same theorem holds for (2), if we put the following restrictions† on $\lambda_n=\alpha_n+\beta_n i$: $\alpha_n\to+\infty$, $\beta_n/\alpha_n\to 0$. In case λ_n approaches a limit, the region of convergence of (2) is bounded by a circle.‡

Pincherle§ considered (2), not requiring that $\beta_n/\alpha_n\to 0$ but only that $|\arg\lambda_n|<\chi<\pi/2$ and that $\sum_{n=1}^{\infty}1/|\lambda_n|$ diverge and $\sum_{n=1}^{\infty}1/|\lambda_n|^2$ converge. He proves that if (2) converges when $z=z_0$ it converges at all points z such that $|\arg(z-z_0)|\leqq\pi/2-\chi$. He does not argue that this exhausts the region of convergence.

However, as uniform convergence is of paramount interest when we consider series, I shall pass to theorems on uniform convergence.

Landau (loc. cit.) proved that (2) with λ_n real, $\lambda_n\to+\infty$, converges uniformly over any finite region interior to its half-plane of convergence. Writing in the Acta Mathematica, Nörlund (loc. cit.) proved that (1) converges uniformly over any half-plane interior to its half-plane of convergence and lying to the right of the axis of imaginaries, that is, if (1) converges when $z=x_0>0$ it converges uniformly when $R(z)\geqq x_0+\delta$, $\delta>0$. In his book‖ he states this same theorem without proof and does not mention the restriction that the half-plane of uniform convergence lie to the right of the axis of imaginaries. The method employed by Nörlund in the Acta Mathematica is by a transformation which seems particular in character and not

* This Bulletin, vol. 34 (1928), p. 259.

† Schnee, Berliner Dissertation, Göttingen, 1908, p. 74 ff.

‡ Jensen, Tidsskrift for Mathematik, 1884, p. 63–72.

§ Palermo Rendiconti, vol. 37 (1914), p. 379.

‖ *Differenzenrechnung*, p. 258.

to apply to the more general series (2). Landau proved his theorem by a lemma which on account of its great importance in the theory of factorial series I shall now state. By means of it all known theorems on the convergence of factorial series can be proved, as well as the fundamental convergence theorems for Dirichlet series, power series, etc. This lemma was proved in various forms by Dedekind,* du Bois Reymond† and Cahen.‡ To avoid repetition it is stated in general form for uniform convergence.

LEMMA 1. *Hypotheses*: (i) $a_n(z)$ *and* $b_n(z)$ *are defined at all points of a set* R; (ii) $\sum_{n=1}^{\infty} a_n(z)$ *converges uniformly over* R; (iii) *there exists a fixed* m *and* M *such that* $|b_m(z)| + \sum_{n=m}^{\infty} |\Delta b_n(z)| < M$ *for all* z*'s of* R. *Conclusion*: $\sum_{n=1}^{\infty} a_n(z)b_n(z)$ *converges uniformly over* R.

By means of Lemma 1, I have been able to establish more general theorems on uniform convergence than were previously known.

In a recent paper,§ I proved the following theorem with reference to series (2).

Restrict $\lambda_n = \alpha_n + \beta_n i$ *as follows. Require that* $\alpha_n \to +\infty$, $\sum_{n=1}^{\infty} (1/\lambda_n)$ *diverge and that* $\sum_{n=1}^{\infty} (\beta_n/\alpha_n)^2$ *converge. Under these restrictions, if* (2) *converges when* $z = z_0 = x_0 + y_0 i$, *it converges uniformly over any angular region defined by* $x \geqq x_0$ *and* $|y - y_0| \leqq a(x - x_0)^m$, m *any positive integer and* a *any positive number.*

Under these same restrictions on λ_n, I am now able not only to extend Nörlund's theorem of the half-plane of uniform convergence to series (2), which I was not able to do at that time, but to rid the theorem of the unnatural restriction that the half-plane of uniform convergence lie wholly to the right of the axis of imaginaries.

Let us suppose the series to converge when $z = z_0 = x_0 + y_0 i$.

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* *Zahlentheorie*, 2d ed., p. 373.

† *Neue Lehrsatz über die Summen unendlicher Reihen*, Auftrittsprogramm, 1870, p. 10.

‡ Annales de l'Ecole Normale Supérieure, vol. 8, p. 79.

§ Transactions of this Society, vol. 31 (1929), p. 233.

Let $z_1 = x_1 + y_1 i$ be fixed and such that $x_1 > x_0$, and let $z' = x' + y' i$ be a variable, and require that $x' \geqq x_1$. Let

$$a_n(z) = c_n \frac{\lambda_1 \cdots \lambda_n}{(z_0 + \lambda_1) \cdots (z_0 + \lambda_n)},$$

and

$$b_n(z_0, z) = \frac{(z_0 + \lambda_1) \cdots (z_0 + \lambda_n)}{(z + \lambda_1) \cdots (z + \lambda_n)}.$$

We find that

$$\frac{\Delta b_n(z_0, z')}{\Delta b_n(z_0, z_1)} = \frac{z_0 - z'}{z_0 - z_1} \cdot \frac{z_1 + \lambda_1}{z' + \lambda_1} \left[\frac{(z_1 + \lambda_2) \cdots (z_1 + \lambda_{n+1})}{(z' + \lambda_2) \cdots (z' + \lambda_{n+1})} \right],$$

but $[(z_0 - z')/(z_0 - z_1)][(z_1 + \lambda_1)/(z' + \lambda_1)]$, as a function of z', remains finite over the half-plane $x' \geqq x_1$. The expression in the bracket is $b_n(z_1, z')$ if the series $\lambda_1, \cdots, \lambda_n$ be replaced by $\lambda_2, \cdots, \lambda_{n+1}$. I proved in the Transactions paper just referred to that this remains uniformly finite over the region in question from which neighborhoods of those points $-\lambda_2, -\lambda_3, \cdots$ which lie in it have been removed. Hence, owing to the convergence of $\sum_{n=1}^{\infty} |\Delta b_n(z_0, z_1)|$, as also established in my Transactions paper, we see that $\sum_{n=1}^{\infty} |\Delta b_n(z_0, z')|$ converges uniformly over the half-plane $x' \geqq x_1$. Moreover, $b_n(z_0, z')$ remains finite over this half-plane. Hence, by the lemma, (2) converges uniformly over the half-plane $x \geqq x_1$ and Nörlund's theorem is extended to this more general series.

This theorem with the theorem quoted from my paper of last April seems to settle in a reasonably satisfactory way the question of uniform convergence for (1), and (2) with λ_n restricted as indicated. It is to be noticed that the last theorem does not entirely replace the first.

The restriction that $\sum_{n=1}^{\infty} (1/\lambda_n)$ diverge is a natural one, for in the contrary case it can be readily proved* that (2) either converges at all points or diverges at all points. The points $-\lambda_1, -\lambda_2, \cdots$ are, of course, excepted here as elsewhere in the paper.

If we do not impose the restriction that $\sum_{n=1}^{\infty} (\beta_n/\alpha_n)^2$ converge but in its place only require that $\beta_n/\alpha_n \to 0$, the best that I have been able to do for a region of uniform convergence is the result given in my paper of last April, namely, the familiar angular region determined by $x \geqq x_0$ and $|y - y_0| \leqq a(x - x_0)$.

* Schnee, loc. cit.

3. *Summability.* The summability according to the Cesàro-Hölder method of series (1), that is, the simple factorial series, was treated by Bohr.* His method of proof is by a generalization of Lemma 1. He finds that there exists an abscissa of summability analogous to the abscissa of convergence and he proves the series uniformly summable over any finite region within its corresponding half-plane of summability. A somewhat similar line of attack can be applied to the more general series (2) for summability by Riesz weighted means. I proceed to do this.

I define summation by Riesz weighted means as follows. Given $\sum_{n=0}^{\infty} a_n$ and $\lambda_1, \lambda_2, \cdots$. Let $\sigma_n = \sum_{n=1}^{n}(1/\lambda_n)$ and $s_n = \sum_{n=0} a_n$; and let

$$s_n^{(1)} = \frac{1}{\sigma_{n+1}}\left[\frac{1}{\lambda_1}s_0 + \frac{1}{\lambda_2}s_1 + \cdots + \frac{1}{\lambda_{n+1}}s_n\right].$$

If $s_n^{(1)}$ approaches a limit, that limit will be called the sum of order 1 of $\sum_{n=1}^{\infty} a_n$ by Riesz means weighted according to the sequence $1/\lambda_n$. The method clearly can be iterated to the rth order. I treat first only the sum of order 1. To do this I generalize Lemma 1.

LEMMA 2. *Hypotheses*: (i) *$a_n(z)$ and $b_n(z)$ are defined at all points of a set R.*

(ii) *The sum*

$$s_n^{(1)}(z) = \frac{1}{\sigma_{n+1}} \sum_{n=0}^{n} \frac{1}{\lambda_{n+1}} \sum_{n=0}^{n} a_n(z)$$

approaches a limit uniformly over R when $n \to \infty$.

(iii) *The quantity*

$$\begin{aligned} &|b_{n+2}(z)| + \frac{1}{|\sigma_{n+1}|} \sum_{n=0}^{n} |\Delta b_{n+1}(z)|\,|\sigma_{n+1}| \\ &+ \frac{1}{|\sigma_{n+1}|} \sum_{n=0}^{n} \left|\frac{\lambda_{n+2}}{\lambda_{n+1}}\right| |\Delta b_{n+1}(z)|\,|\sigma_{n+1}| \\ &+ \frac{1}{|\sigma_{n+1}|} \sum_{n=0}^{n} \frac{1}{|\lambda_{n+1}|} \sum_{n=0}^{n} |\Delta\lambda_{n+1}\Delta b_n(z)|\,|\sigma_{n+1}| \end{aligned} \tag{3}$$

remains uniformly finite over R.

* Göttinger Nachrichten, 1909, p. 247.

Conclusion: $\sum_{n=0}^{\infty} a_n(z)b_n(z)$ *is uniformly summable of order* 1 *over R by this method.*

This lemma is readily proved if we apply summation by parts twice to

$$\frac{1}{\sigma_{n+1}} \sum_{n=0}^{n} \frac{1}{\lambda_{n+1}} \sum_{n=0}^{n} a_n(z)b_n(z).$$

It will be applied to series (2), where we again impose the restrictions on $\lambda_n=\alpha_n+\beta_n i$, $\alpha_n\to+\infty$, $\sum_{n=1}^{\infty}(1/\lambda_n)$ diverges and $\sum_{n=1}^{\infty}(\beta_n/\alpha_n)^2$ converges. I have proved the following theorem under these hypotheses.

If (2) *is summable of order* 1 *at* $z_0=x_0+y_0 i$, *it is uniformly summable of order* 1 *throughout an angular region defined by* $x\geqq x_0$ *and* $|y-y_0|\leqq m(x-x_0)$, *where m is any positive constant.*

The principal steps only in the proof will be given.

We assume $x_0>0$ and $\alpha_1>0$ which entails no loss of generality. Let $b_0(z)=1$, and when $n>0$,

$$b_n(z) = b_n(z_0,z) = \frac{(z_0+\lambda_1)\cdots(z_0+\lambda_n)}{(z+\lambda_1)\cdots(z+\lambda_n)}.$$

Let $a_0(z)=c_0$, and when $n>0$,

$$a_n(z) = c_n \frac{\lambda_1\cdots\lambda_n}{(z_0+\lambda_1)\cdots(z_0+\lambda_n)}.$$

Substitute in formula (3). We wish to prove that the resulting expression remains less than a constant. To show this we take the ratio of each summand in (3) to the corresponding summand with z replaced by its real part x, and λ_n replaced by its real part α_n, and σ_{n+1} by $\tau_{n+1}=\sum_{n=1}^{n+1}(1/\alpha_n)$. Each of these ratios remains less than a constant. I refer to Preliminary Theorems 1 and 2 of my own paper (Transactions of this Society, vol. 31 (1929), p. 233), only remarking that

$$\Delta\lambda_{n+1}\Delta b_n(z_0,z) = (z_0-z)\Delta b_{n+1}(z_0,z).$$

Moreover we have

$$
\begin{aligned}
|\, b_{n+2}(x_0,x)\,| &+ \frac{1}{\tau_{n+1}} \sum_{n=0}^{n} |\, \Delta b_{n+1}(x_0,x)\,|\, \tau_{n+1} \\
&+ \frac{1}{\tau_{n+1}} \sum_{n=0}^{n} \frac{\alpha_{n+2}}{\alpha_{n+1}} |\, \Delta b_{n+1}(x_0,x)\,|\, \tau_{n+1} \\
&+ \frac{1}{\tau_{n+1}} \sum_{n=0}^{n} \frac{1}{\alpha_{n+1}} \sum_{n=0}^{n} |\, \Delta\alpha_{n+1}\Delta b_n(x_0,x)\,|\, \tau_{n+1} \\
= b_{n+2}(x_0,x) &- \frac{1}{\tau_{n+1}} \sum_{n=0}^{n} \Delta b_{n+1}(x_0,x)\tau_{n+1} \\
&- \frac{1}{\tau_{n+1}} \sum_{n=0}^{n} \frac{\alpha_{n+2}}{\alpha_{n+1}} \Delta b_{n+1}(x_0,x)\tau_{n+1} \\
&+ \frac{1}{\tau_{n+1}} \sum_{n=0}^{n} \frac{1}{\alpha_{n+1}} \Delta\alpha_{n+1}\Delta b_n(x_0,x)\tau_{n+1} \\
= \frac{1}{\tau_{n+1}} \sum_{n=0}^{n} \frac{1}{\alpha_{n+1}} &\sum_{n=0}^{n} k_n b_n(x_0,x), \qquad (k_0 = 1, \quad k_n = 0),
\end{aligned}
$$

when $n>0$; that is, the k_n's are the coefficients of that series (2) for which $s^{(1)}(x_0)\equiv 1$, which is the series for which

$$
s_n^{(1)}(x) \equiv 1\,; \quad \frac{1}{\tau_{n+1}} \sum_{n=0}^{n} \frac{1}{\alpha_{n+1}} \sum_{n=0}^{n} k_n b_n(x_0,x) = 1\,;
$$

and our theorem follows by Lemma 2.

There is a half-plane of summability in every way analogous to the half-plane of convergence. This theorem follows from the preceding as in the corresponding case for convergence.

The generalization of Lemma 2 to a weighted sum of any fixed number of iterations is straightforward. The formulas, however, are increasingly complicated. When we come to apply these lemmas to factorial series, I find a difficulty in proving that the ratio of each summand in z and λ_n to the corresponding summand in x and α_n remains uniformly finite. I find that I must assume in addition to the assumption already made that λ_{n+1}/λ_n remains finite and that $\Delta(\lambda_{n+1}/\lambda_n)\to 0$ and that $x-x_0>\delta>0$. I am then able to establish theorems for rth order summation by weighted means analogous to those just proved for first order summation by weighted means, namely uniform summation over an angular region bounded by straight lines

from which the half-plane of summation follows. The additional assumptions on λ_n and the assumption $(x-x_0)>\delta$ are probably incident to the method of proof. Lines of summability follow immediately.

It is to be noticed, however, that all these results are more general than those of Bohr (loc. cit.) even for the simple series (1) which he treats and the Cesàro-Hölder method which he uses. He only proves uniform summability over a finite region and he retains the restriction $x-x_0>\delta$.

4. *Equiconvergence Theorems.* Landau (loc. cit.) proved in his classical memoir that series (1) and the ordinary Dirichlet series

$$\sum_{n=1}^{\infty} c_n n^{-z}, \tag{4}$$

where the coefficients are the same, converge at the same points. Kluyver* had already proved a similar theorem for absolute convergence.

In the Transactions of this Society for January, 1922, I set up the series

$$\sum_{n=1}^{\infty} c_n \frac{k}{z+k-1} \frac{\Gamma(nk)}{\Gamma(z+nk)} \cdot \frac{\Gamma(z+k)}{\Gamma(k)}. \tag{5}$$

This series reduces to the ordinary factorial series for $k=1$ and has the ordinary Dirichlet series (4) as a limiting series when $k\to\infty$. It is found that by placing only slight restriction on k we have a wide class of series including the limiting case (Dirichlet series) for which the equiconvergence theorem holds. That is, (5) converges at the same points for the values of k including the limiting case where $k\to\infty$.

Pincherle (loc. cit.) proved (2) and

$$\sum_{n=1}^{\infty} c_n e^{-\sigma_n z}, \quad \sigma_n = \sum_{n=1}^{n} \frac{1}{\lambda_n},$$

equiconvergent provided, among other restrictions, $\sum_{n=1}^{\infty}(1/|\lambda_n|)^2$, converges. In my paper in the Transactions of April, I removed all restrictions on λ_n except that $\sum_{n=1}^{\infty}(1/\lambda_n)^{s+1}$ be convergent and proved an equiconvergence theorem for (2) and for the series $\sum_{n=1}^{\infty} c_n e^{-\sigma_{1n}}$, where

* Nieuw Archiv, (2), vol. 4 (1899), p. 74.

$$g_{1n} = \left(\sum_{n=1}^{n} \frac{1}{\lambda_n}\right)z - \left(\sum_{n=1}^{n} \frac{1}{\lambda_n^2}\right)z^2 + \cdots + (-1)^{s-1}\left(\sum_{n=1}^{n} \frac{1}{\lambda_n^s}\right)z^s .$$

In another theorem in the same paper I proved that, if λ_n is real, $\lambda_n \to +\infty$ and $\sum_{n=1}^{\infty}(1/\lambda_n)$ diverges, then (2) and the Dirichlet series, $\sum_{n=1}^{\infty} c_n e^{-\sigma_n z}, \sigma_n = \sum_{n=1}^{n}(1/\lambda_n)$, have the same line of convergence. The location of the lines of convergence is now given by the familiar formulas for Dirichlet series.

Equisummability theorems analogous to the above for summation by weighted means hold at least if we put the restrictions of §4 on λ_n. Bohr (loc. cit.) has results for the simple factorial series (1) and the ordinary Dirichlet series using the Cesàro-Hölder summation method.

5. *Function Represented.* The question of the nature of the function representable by a class of series and how to obtain this representation is one of the most interesting questions in the study of series.

I have already remarked that if $\Omega(z)$ is the function represented by (1), then

$$\Delta\Omega(z) = -\sum_{n=1}^{\infty} c_n \frac{n!}{z(z+1)\cdots(z+n)} .$$

In order for the series to be of much use in the study of difference equations it is necessary to prove other theorems relative to algebraic operations with factorial series and to uniqueness of representation in factorial series. These necessary theorems have been handled principally by Nörlund (loc. cit.). The uniqueness theorem, so important in work of this kind, does not hold for the binomial coefficient series, as was shown by Pincherle,* and this fact robs that series of much interest which attaches to factorial series. It would be interesting to investigate the situation as regards these useful theorems for series (2). This does not seem to have been done.

That $\Omega(z)$ is analytic over a region of uniform convergence is immediate. This does not include the point ∞, of course, which is not within the region of uniform convergence. There are poles at each of the points $-\lambda_1, -\lambda_2, \cdots$, which lie within the region of convergence. The line of convergence differs from

* Atti, Accademia dei Lincei, Rendiconti, (5), vol. 2, pp. 417–426.

the circle of convergence of a power series in that no singular point need necessarily lie upon it. As a matter of fact Nörlund (loc. cit., p. 374) has shown that if we consider series (1′) there is a line, $x=l$, which is approached by a, the abscissa of convergence, when $\omega\to 0$. We write $a(\omega)$. This line is characteristic of the function which ceases to be analytic at all points, or at least does not remain finite, within the band $l-\epsilon\leqq x\leqq l+\epsilon$ no matter how small $\epsilon>0$ may be.

Nörlund has shown (loc. cit., p. 347) that the behavior of Ω as $z\to\infty$ remaining in the half-plane of convergence is given by the following formulas:

$$\lim \Omega(z) = c_0,$$
$$\lim z\Omega(z) = c_1,$$
$$\lim (z+1)\left[z\Omega(z) - c_1\right] = c_2,$$
$$\lim (z+2)\left[(z+1)\left\{z\Omega(z) - c_1\right\}\right] = c_3,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

These relations seem to give a proper description of the asymptotic behavior of the function.

Watson* has written a long paper in which his thesis is to prove that functions asymptotic to divergent power series in the sense of Poincaré,

$$f(x) \sim a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots,$$

can be represented by a convergent factorial series. He succeeds in establishing some very general theorems. These theorems are a little long in statement and are not given in detail.

Nörlund (loc. cit., p. 379) proves that functions represented by divergent power series of the form

$$a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots,$$

summed by Borel's integral definition can be represented by convergent factorial series.

Necessary and sufficient conditions that a function be repre-

* Palermo Rendiconti, vol. 34 (1912), p. 41.

sentable by convergent factorial series have* been obtained. They center around Laplace's integral

$$\int_0^1 Q(t)t^{x-1}dt.$$

6. *Difference Equations.* With the background of these theorems the establishment of a broad theory of analytic solutions of linear difference equations is no very difficult matter. This has been done by Nörlund in a most excellent way. The essential preliminary, however, was the satisfactory theory of factorial series. Nörlund's principal theorem on linear difference equations, in a word, is that linear difference equations whose coefficients can be developed into factorial series possess a fundamental system of analytic solutions which can be expressed in terms of factorial series. For exact statement of the theorem I refer to his book.† The results may also be found in his papers. This is, so far as I know, really the most important application of factorial series.

Solutions of linear differential equations under certain quite general conditions can be written as factorial series,‡ which form has been useful in studying them in the neighborhood of singular points. This has not seemed to me, however, as interesting or natural as the difference equation results.

7. *Work of Carmichael.* I wish now to spend a short while discussing in greater detail the work of R. D. Carmichael (loc. cit., (2)). The researches of which I wish to speak are embodied in a series of papers beginning with the Transactions of this Society for 1916 and including three papers in the American Journal of Mathematics, and one in the Bulletin of the American Mathematical Society. This work is in the nature of a generalization but is peculiarly complete. The papers seem to have come from the asymptotic notions of Poincaré particularly as used by Birkhoff, Carmichael, and others in the study of difference equations. Lemma 1 of this paper is abundantly used. The author acknowledges his indebtedness to Landau.

* See Encyklopädie, vol. II–C–1–12, p. 1272; also Nörlund, loc. cit., p. 371.

† *Differenzenrechnung*, Berlin, 1924, p. 365.

‡ Horn, Mathematische Annalen, vol. 71, p. 510; H. T. Davis tells me that he has found factorial series indispensible in his studies of differential equations of infinite order.

In Landau's paper the part of b_n of our Lemma 1 is played by

$$\frac{z_0(z_0+1)\cdots(z_0+n-1)}{z(z+1)\cdots(z+n-1)}$$

which he shows to be asymptotic to $n^{-(z-z_0)}$. Carmichael sets up series for which

$$b_n \sim n^{\mu(z-z_0)}\left(t_0+\frac{t_1}{n}+\cdots\right). \tag{6}$$

It is to be noticed that this is a more general asymptotic form for b_n. This form is not featured by the author but seems to me the central formula of the papers and may have been the point of beginning of the research. If so, the problem would immediately be to generalize

$$\frac{(n-1)!}{z(z+1)\cdots(z+n-1)}$$

to forms which yield this asymptotic representation for the function b_n. Carmichael assumes a function

$$g(z) \sim z^{P(z)}e^{Q(z)}\left(1+\frac{a_1}{z}+\frac{a_2}{z^2}+\cdots\right)$$

analytic at least when z is in a sector including the axis of reals and sufficiently large in absolute value. $P(z)$ and $Q(z)$ are polynomials. The establishment of asymptotic form (6) is now straightforward. At some points more stringent restrictions are put on $g(z)$. The existence of such functions is readily established. Carmichael's papers concern themselves with series of the type $\sum_{n=0}^{\infty} c_n\, g(z+n)$ and closely allied forms. The theory runs much as would be expected and much like the theory of factorial series. There is a half-plane of convergence, of absolute convergence, etc. with bounding lines given by expected familiar formulas. He does not treat summability of the series when divergent. However, Professor Carmichael tells me that this has been treated according to the Cesàro-Hölder method by C. F. Green in a dissertation at the University of Illinois which has not been published. Carmichael makes considerable progress on the representation of functions by such series. He proves theorems showing that functions given

by asymptotic power series or such series at most summable by Borel's integral method can be represented by convergent series of his functions. For exact statements of results I refer to Carmichael's papers. The asymptotic form of the function represented by such series is investigated. Again it is what we expect from known facts relative to factorial series. Of particular significance it seems to me is the author's definition of an asymptotic series using the series of his papers. It is a considerable generalization of the notion of Poincaré. More recently Ritt* has defined and studied asymptotic Dirichlet series.

Carmichael is also led to study series of the type

$$\sum_{n=1}^{\infty} C_n e^{-\lambda_n^{(1)} z - \lambda_n^{(2)} z^2 - \cdots - \lambda_n^{(k)} z^k},$$

the first time, so far as I know, that such series have been studied. He greatly restricts his parameters but even then obtains interesting results. The series is introduced only for its immediate use and the author states that he has developed a more extensive theory than he publishes.

8. *Conclusion.* In conclusion, numerous mathematicians have written on factorial and allied series. I regard the work of Nörlund as most significant. Next that of Landau followed by that of Pincherle and Carmichael. One should also surely mention Jensen, Bendixon, Nielson and Schnee. There have been other writers. No effort has been made to be exhaustive. My apologies are offered to those authors whose fine papers are not mentioned. Landau did not start the study of factorial series nor did he originate many of the theorems. He, however, builds a consistent theory, proving all theorems with the utmost rigor and writing with his peculiarly clear and elegant style. Nörlund takes up Landau's task and develops a more or less complete theory for the simple factorial series and what he proves to be its equivalent in many ways, namely series (1′). His interest is, probably, primarily in difference equations and his theory of factorial series lends itself well to that study. Pincherle adds a number of theorems. He is primarily concerned with series (2). I have just spoken of Carmichael's work.

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* American Journal of Mathematics, vol. 50 (1928), pp. 73–85.

HAUSDORFF'S THEOREM CONCERNING HERMITIAN FORMS

BY M. H. STONE

In 1919, Hausdorff proved an interesting theorem to the effect that the complex values assumed by the Hermitian form

$$\sum_{\alpha,\beta=1}^{n} a_{\alpha\beta}\bar{x}_\alpha x_\beta,$$

when the complex numbers $x_1, \cdots, x_n$ take on all values compatible with the relation

$$\sum_{\alpha=1}^{n} \bar{x}_\alpha x_\alpha = 1,$$

constitute a convex point set in the complex plane.* In his demonstration Hausdorff employs the transformation of Hermitian symmetric forms of this type to principal axes. It is obvious, therefore, that the extension of this method to more general instances is impossible, because the principal axis transformation may no longer be available, a situation which arises, for instance, in the important case of unbounded forms in infinitely many variables. On this account, a proof along less special lines is of considerable interest. It is gratifying to ascertain that an entirely elementary, explicit, and general demonstration of Hausdorff's theorem can be devised by making it depend upon the theorem for binary forms. The proof we shall give involves nothing more difficult than manipulations of complex numbers and the solution of quadratic equations.

We first indicate what we shall mean, abstractly, by an Hermitian bilinear form. Let $\mathcal{J}$ be a class of elements in which the operations $+$ (vector addition) and $a\cdot$ (scalar multiplication by an arbitrary complex constant a) are significant and possess their customary algebraic properties; let $\mathcal{J}$ be closed under these operations. Such a class may be called a complex vector space, its elements complex vectors. A complex-valued

* Mathematische Zeitschrift, vol. 3 (1919), pp. 314–316.

function $B(f, g)$ defined for every pair of elements (f, g) in $\mathfrak{F}$ is said to be an Hermitian bilinear form if and only if it has the following properties:

$$B(af,g) = aB(f,g),$$
$$B(f_1 + f_2,g) = B(f_1,g) + B(f_2,g),$$
$$B(f,ag) = \bar{a}B(f,g),$$
$$B(f,g_1 + g_2) = B(f,g_1) + B(f,g_2).$$

Such a form is said to be Hermitian symmetric and positive definite if and only if $B(f, g)$ and $B(g, f)$ are conjugate complex numbers and the real number $B(f, f)$ is never negative, vanishing only when f is the null element or null vector. When B is Hermitian symmetric and positive definite, we can establish the Cauchy-Schwarz inequality

$$|B(f,g)| \leqq [B(f,f)B(g,g)]^{1/2},$$

by the usual considerations.

We assume that there exist in $\mathfrak{F}$ two Hermitian bilinear forms B_1 and B_2, the second of which is Hermitian symmetric and positive definite. Let W be the set of complex values assumed by B_1 when B_2 is required to have the value 1. We wish to show that W is a convex set; that is, that, whenever z_1 and z_2 are in W, the points of the line segment joining them are also in W.

If W contains just one point, there is nothing to prove, so that this case may be ruled out. Now, let z_1 and z_2 be distinct complex numbers in W, and f_1 and f_2 elements or vectors in $\mathfrak{F}$ such that

$$B_1(f_1,f_1) = z_1, \quad B_1(f_2,f_2) = z_2, \quad B_2(f_1,f_1) = 1, \quad B_2(f_2,f_2) = 1.$$

We consider the binary Hermitian forms

$$\begin{aligned} A_1(x_1,x_2) &= B_1(x_1f_1 + x_2f_2, x_1f_1 + x_2f_2) \\ &= z_1\bar{x}_1x_1 + B_1(f_2,f_1)\bar{x}_1x_2 + B_1(f_1,f_2)x_1\bar{x}_2 + z_2x_2\bar{x}_2, \\ A_2(x_1,x_2) &= B_2(x_1f_1 + x_2f_2, x_1f_1 + x_2f_2) \\ &= \bar{x}_1x_1 + B_2(f_2,f_1)\bar{x}_1x_2 + B_2(f_1,f_2)x_1\bar{x}_2 + x_2\bar{x}_2. \end{aligned}$$

We must show that, when A_2 is restricted to have the value 1,

A_1 takes on all values represented by points of the line segment joining z_1 and z_2.

It is convenient to introduce a new binary form

$$A(x_1, x_2) = (A_1 - z_2A_2)/(z_1 - z_2) = \bar{x}_1x_1 + a_{12}\bar{x}_1x_2 + a_{21}x_1\bar{x}_2.$$

In order that A_1 should exhibit the requisite behavior, it is necessary and sufficient that, while $A_2=1$, A should take on all real values from 0 to 1 inclusive. We shall now find values for x_1 and x_2 which bring about the desired result. We first define a complex number γ as follows: $\gamma=\pm 1$ when $\bar{a}_{12}=a_{21}$, and $\gamma=\pm(\bar{a}_{12}-a_{21})/|\bar{a}_{12}-a_{21}|$ when $\bar{a}_{12}\neq a_{21}$, the plus or the minus sign being chosen in each case so as to render the real number $\beta=R(\gamma B_2(f_2, f_1))$ not negative. We set $x_1=x$, $x_2=\gamma y$, where x and y are real variables, and substitute these values in the expressions for A and A_2, obtaining

$$A = x^2 + \alpha xy,$$

$$\alpha = \bar{\alpha} = \pm(a_{12} + a_{21}) \quad \text{or} \quad \pm(a_{12}\bar{a}_{12} - a_{21}\bar{a}_{21})/|\bar{a}_{12} - a_{21}|,$$

$$A_2 = x^2 + 2\beta xy + y^2.$$

We observe that $0 \leqq \beta \leqq 1$, in view of the inequalities

$$|\beta| = |R(\gamma B_2(f_2,f_1))| \leqq |\gamma|\,|B_2(f_2,f_1)|$$
$$= |B_2(f_2,f_1)| \leqq [B_2(f_1,f_1)B_2(f_2,f_2)]^{1/2} = 1.$$

Hence the equation $A_2=1$ can be satisfied by taking $y=-\beta x+[1-(1-\beta^2)x^2]^{1/2}$, and, with this value for y, the form A becomes

$$A = (1 - \alpha\beta)x^2 + \alpha x[1 - (1 - \beta^2)x^2]^{1/2}.$$

For $x=0$, A takes on the value 0, for $x=1$ the value 1; since A is a continuous real-valued function of x, it takes on at least once every real value from zero to one when x varies between these same values, as we wished to show.

This completes the demonstration of Hausdorff's theorem that W is a convex set.

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MATRICES WHOSE CHARACTERISTIC EQUATIONS ARE CYCLIC*

BY T. A. PIERCE

One of Sylvester's theorems† on matrices states that if the characteristic equation

$$|M - \lambda I| = f(\lambda) = 0 \tag{1}$$

of a square matrix M has the roots $\lambda_1, \lambda_2, \cdots, \lambda_n$, then the characteristic equation

$$|\phi(M) - \rho I| = g(\rho) = 0 \tag{2}$$

of any integral function of M, namely, $\phi(M)$, has the roots $\rho_i = \phi(\lambda_i)$, $i = 1, 2, \cdots, n$. In this note an isomorphism is shown to exist between the algebraic and matric roots of (1) when this equation is cyclic. Certain consequences of this isomorphism are given. Since

$$\begin{pmatrix} -a & -b & \cdots & -k & -l \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

is the matrix which has $\lambda^n + a\lambda^{n-1} + b\lambda^{n-2} + \cdots + k\lambda + l = 0$ as its characteristic equation,‡ Sylvester's theorem furnishes a method of effecting the Tschirnhaus transformation $\rho = \phi(\lambda)$ on any equation. When (1) is cyclic an especially interesting type of Tschirnhaus transformation is possible.

Suppose (1) to be a cyclic equation of degree n with the relations $\lambda_{i+1} = \phi(\lambda_i)$, $i = 1, 2, \cdots, n$ and $\lambda_{n+j} = \lambda_j$ connecting its roots. For simplicity of notation let ϕ_i be the ith iterated function of ϕ so that $\lambda_2 = \phi(\lambda_1) = \phi_1(\lambda_1)$, $\lambda_3 = \phi(\lambda_2) = \phi(\phi(\lambda_1)) = \phi_2(\lambda_1)$, and in general $\lambda_{i+1} = \phi_i(\lambda_1)$. By Sylvester's theorem the roots of (2) then are

* Presented to the Society, December 30, 1929.

† Sylvester, *Mathematical Papers*, vol. 4, p. 133; Frobenius, Journal für Mathematik, vol. 84, p. 11; Dickson, *Algebren und ihre Zahlentheorie*, p. 18.

‡ Wedderburn, Annals of Mathematics, vol. 27 (1926), p. 247.

$\rho_1 = \phi(\lambda_1) = \lambda_2,\ \rho_2 = \phi(\lambda_2) = \lambda_3, \cdots, \rho_n = \phi(\lambda_n) = \lambda_{n+1} = \lambda_1.$

Thus the roots of (1) and (2) are the same, and M and $\phi(M)$ have the same characteristic equation. Similarly, if we form the characteristic equation of $\phi_i(M)$, we find that its roots are

$$\rho_1 = \phi_i(\lambda_1) = \lambda_{i+1}, \quad \rho_2 = \phi_i(\lambda_2) = \lambda_{i+2}, \cdots, \rho_n = \phi_i(\lambda_n) = \lambda_i,$$

where the subscripts are to be reduced modulo n. Hence we have the following theorem.

THEOREM 1. *If the characteristic equation, $f(\lambda)=0$, of the matrix M is cyclic with the relations $\lambda_i=\phi_i(\lambda_1)$, $i=1, 2, \cdots, n$, connecting its roots, then the matrices $M, \phi_1(M), \cdots, \phi_{n-1}(M)$, all have the same characteristic equation.*

Applying the Hamilton-Cayley theorem we have the following additional theorem.

THEOREM 2. *Under the conditions of Theorem* 1 *above $f(\lambda)=0$ has its full complement of matric roots, namely,*

$$M, \phi_1(M), \cdots, \phi_{n-1}(M).$$

The matric roots of this last theorem may be called conjugate* and may be denoted by

$$M_1, M_2 = \phi_1(M_1), \cdots, M_n = \phi_{n-1}(M_1). \tag{3}$$

Since these matrices are all functions of one matrix they are commutative under multiplication and division, and since the same cyclic relations exist among the M's as among the λ's, we have the following theorem.

THEOREM 3. *The field generated by the roots of any cyclic equation is isomorphic to the field generated by the matric roots of this same equation.*

Consider now any integral function of the matrices $M_1, \cdots, M_n$, namely $F(M_1, \cdots, M_n)$, and form its characteristic equation

$$\left| F(M_1, \cdots, M_n) - \rho I \right| = 0. \tag{4}$$

Now $F(M_1, \cdots, M_n)=G(M_1)$ by virtue of (3), and hence the roots of (4) are by Sylvester's theorem $\rho_i=G(\lambda_i)$, $i=1, 2, \cdots, n$. But

* Taber, American Journal of Mathematics, vol. 13, p. 159.

$$\begin{aligned}\rho_1 = G(\lambda_1) &= F[\lambda_1, \phi_1(\lambda_1), \cdots, \phi_{n-1}(\lambda_1)] \\ &= F[\lambda_1, \lambda_2, \cdots, \lambda_n], \\ \rho_2 = G(\lambda_2) &= F[\lambda_2, \phi_1(\lambda_2), \cdots, \phi_{n-1}(\lambda_2)] \\ &= F[\lambda_2, \lambda_3, \cdots, \lambda_1],\end{aligned}$$

and in general we find that

$$\rho_i = G(\lambda_i) = F[\lambda_i, \lambda_{i+1}, \cdots, \lambda_{i-1}];$$

thus we obtain the theorem which follows:

THEOREM 4. *Let the characteristic equation of M_1 be cyclic and have the roots $\lambda_1, \lambda_2, \cdots, \lambda_n$ and the corresponding matric roots $M_1, M_2, \cdots, M_n$. Then, if $F(M_1, \cdots, M_n)$ be an integral function, the roots of the characteristic equation of $F(M_1, \cdots, M_n)$ are $F(\lambda_1, \cdots, \lambda_n)$ and the quantities obtained from this by permuting $\lambda_1, \lambda_2, \cdots, \lambda_n$ cyclically. The matric roots of this equation are obtained from $F(M_1, \cdots, M_n)$ by permuting $M_1, M_2, \cdots, M_n$ cyclically. Moreover this equation is cyclic.*

Thus, for example, if $|M_1 - \lambda I| = 0$ is cyclic and has the roots $\lambda_1, \lambda_2, \cdots, \lambda_n$, the roots of $|M_1 + M_2 - \rho I| = 0$ are $\rho_1 = \lambda_1 + \lambda_2$, $\rho_2 = \lambda_2 + \lambda_3, \cdots, \rho_n = \lambda_n + \lambda_1$. To the first equation the Tschirnhaus transformation $\rho = \lambda + \phi(\lambda)$ has been applied and this transformation affects the roots of the equation in pairs.

If C_m denotes the mth compound of the matrix M it is known* that the roots of the characteristic equation

$$|C_m - \rho I| = 0 \tag{5}$$

are the products m at a time of the roots of $|M - \lambda I| = 0$. If the latter is a cyclic equation its roots are all functions of one root and the roots of (5) are therefore all functions of this root. By the isomorphism of Theorem 3 the roots of (5) correspond to matrices all of which are functions of one matrix and hence are commutative under multiplication. Moreover by Theorem 3 these matrices all satisfy (5) so that (5) has its full complement of matric roots. These matric roots are, however, in general of order different from the degree of (5).

THE UNIVERSITY OF NEBRASKA

* Pascal, *Repertorium der höheren Mathematik*, vol. 2, 1910, p. 139; Whittaker, Proceedings of the Edinburgh Mathematical Society, vol. 35 (1916), p. 2.

ON THE DIRECT PRODUCT OF A DIVISION AND A TOTAL MATRIC ALGEBRA*

BY F. S. NOWLAN

This paper establishes certain theorems concerning an algebra A which is expressible as the direct product† of a division algebra D and a total matric algebra M. It is moreover not assumed that D and M are subalgebras of A. We let δ and n^2 represent the orders of D and M respectively. It follows that δn^2 is the order of A. We represent the modulus of A by be where b and e are the respective moduli of D and M. In agreement with the usual notation, we write

$$e = \sum e_{ii}, (i = 1, \cdots, n),$$

where e_{ij}, $(i,j = 1, \cdots, n)$, are the basal units of M.

For the proof of Theorem 1, we express the zero elements of algebras A, D and M by Z, z_d and z_m respectively. Thereafter we employ the symbol 0 without ambiguity. Since the elements of D and M are commutative with each other and a zero element of an algebra is unique, we have‡ $Z = z_d z_m$.

THEOREM 1. *If $dm = Z$, where d and m are any elements of D and M, respectively, then either $d = z_d$ or $m = z_m$.*

For, if $d \neq z_d$, it possesses an inverse d^{-1}. It follows that

$$bm = d^{-1}Z = d^{-1}z_d z_m = z_d z_m = Z.$$

Writing

$$m = \sum_{i,j=1}^{n} \alpha_{ij} e_{ij},$$

we have

$$\sum_{i,j=1}^{n} \alpha_{ij} b e_{ij} = Z.$$

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* Presented to the Society, June 18, 1927.

† Dickson, *Algebras and their Arithmetics*, p. 72.

‡ In the proof, let $Z = z_1 z_2$, where z_1 is in D and z_2 in M. Then

$$Z = Z \cdot z_d z_m = z_1 z_2 \cdot z_d z_m = z_1 z_d \cdot z_2 z_m = z_d z_m.$$

If $m \neq z_m$, there must be some $\alpha_{rs} \neq 0$. Multiplying the equation above on the left by e_{rr} and on the right by e_{ss}, we obtain $\alpha_{rs} b e_{rs} = Z$, whence $b e_{rs} = Z$. Multiplying on the left and right by e_{ir} and e_{si} respectively and summing with respect to i, we obtain

$$b \sum e_{ii} = be = Z,$$

where be is the modulus of A. It follows that $m = z_m$ in case $d \neq z_d$.

THEOREM 2. *If $d_1 m_1 = d_2 m_2$, where d_1 and d_2 are non-zero elements of D, and m_1 and m_2 are non-zero elements of M, then $d_1 = d_2/\alpha$ and $m_1 = \alpha m_2$, where α is a scalar.*

For, let

$$m_1 = \sum_{i,j=1}^{n} \alpha_{ij} e_{ij}, \quad m_2 = \sum_{i,j=1}^{n} \alpha'_{ij} e_{ij}.$$

Then by hypothesis, we have

$$\text{(i)} \qquad \sum_{i,j=1}^{n} (\alpha_{ij} d_1 - \alpha'_{ij} d_2) e_{ij} = 0.$$

Since $m_1 \neq 0$, there is some $\alpha_{rs} \neq 0$. We multiply (i) on the left and right by e_{rr} and e_{ss} respectively, obtaining

$$(\alpha_{rs} d_1 - \alpha'_{rs} d_2) e_{rs} = 0.$$

From Theorem 1, since $e_{rs} \neq 0$, we have

$$d_1 = \frac{\alpha'_{rs}}{\alpha_{rs}} d_2,$$

where $\alpha'_{rs} \neq 0$ since $d_1 \neq 0$. Accordingly $d_1 = d_2/\alpha$, where $\alpha = \alpha_{rs}/\alpha'_{rs}$. Hence $d_1 m_1 = d_2 m_2$ implies $d_1(m_1 - \alpha m_2) = 0$, whence $m_1 = \alpha m_2$ by Theorem 1.

THEOREM 3. *If dm is idempotent in A, then $d = b/\alpha$ and $m = \alpha m'$, where α is a scalar and m' is idempotent in M.*

For, from $d^2 m^2 = dm$, we have $dm^2 = bm$, on multiplying by d^{-1}. From Theorem 2, $d = b/\alpha$ and $m^2 = \alpha m$. Now let $m = \alpha m'$; then

$$m'^2 = \frac{m^2}{\alpha^2} = \frac{\alpha m}{\alpha^2} = \frac{m}{\alpha} = m'.$$

This proves the theorem.

THEOREM 4. *If e' and e'' are any elements of M, then the algebra $e'Ae''$ is the direct product $D\times e'Me''$.*

Since $A=D\times M$, then $e'Ae''=D(e'Me'')$, for e' and e'' are commutative with elements of D.

Let n' and n'' represent the orders of $D(e'Me'')$ and $e'Me''$ respectively. Then* $n'\leqq\delta n''$, where δ is the order of D. In case $e'Me''<M$, we can think of the basal units of M as made up of n'' linearly independent elements of the algebra $e'Me''$ together with certain other n^2-n'' elements in M. Hence the products of these n'' elements of $e'Me''$ by the basal units of D give $\delta n''$ linearly independent elements of $e'Ae''$, since they may be considered as certain of the δn^2 basal units of $A=D\times M$. It follows that $n'=\delta n''$.

DEFINITION. A set of primitive idempotent elements is said to be supplementary in case their sum equals the modulus and if further the product of each pair in either order is zero.

THEOREM 5. *Each algebra e_iMe_j is of order 1, where e_i and e_j belong to a supplementary set of primitive idempotent elements.*

For, since every total matric algebra is simple,† it follows that we may apply the method of §51 of Dickson's *Algebras and their Arithmetics* with M replacing the algebra A. The modulus $\sum e_{ii}$ of M may be written in the form

$$\sum e_{ii}=\sum_{k=1}^{m}e_k,$$

where the $e_1, \cdots, e_m$ are a set of supplementary primitive idempotent elements which include e_i and e_j. This follows from Theorem 3, p. 57 of Dickson's *Algebras and their Arithmetics*, and the last few lines on p. 49 of the same text. We obtain

$$M=\sum_{i,j=1}^{m}M_{ij},$$

where $M_{ij}=e_iMe_j$ and where each of the m^2 algebras M_{ij} is of the same order t. Finally, we are able to write $M=D\times M'$, where M' is a total matric algebra of order m^2. On the other hand we may write $M=(1)\times M$. Since the expression of a simple algebra as the direct product of a division and a total matric algebra is unique‡ in the sense of equivalence, it follows

* Dickson, *Algebras and their Arithmetics*, §18.

† Dickson, *Algebras and their Arithmetics*, p. 80.

‡ Scorza, *Corpi Numerici e Algebre*, 1921, pp. 346–352.

that $M \cong M'$, whence $m=n$. But M was of order tm^2. Accordingly $t=1$ is the order of each of the algebras $M_{ij}=e_iMe_j$.

COROLLARY. *A supplementary set of primitive idempotent elements of a total matric algebra of order n^2 contains exactly n elements.*

THEOREM 6. *If e' is a primitive idempotent element of M, then be' is a primitive idempotent element of $A=D\times M$, and conversely.*

For, consider $e'Ae'$. From Theorem 4, we have $e'Ae' = D\times e'Me'$, where $e'Me'$ may be considered as a total matric algebra of order 1, whose modulus is e'. We must show that be' is the only idempotent of $e'Ae'=be'\cdot A\cdot be'$. We note that any element of $e'Ae'$ may be written in the form $de'me'$, where $d\leqq D$ and $m\leqq M$. If this element is idempotent, then from Theorem 3, replacing A by $e'Ae'$ and M by $e'Me'$, we have $d=b/\alpha$ and $e'me'=\alpha m'$, where α is a scalar and m' is idempotent in $e'Me'$. But $e'Me'$ contains no idempotent other than e'. It follows that $de'me'=be'$ is the only idempotent in $e'Ae'$.

Conversely, let be' be a primitive idempotent element of A and suppose m is any idempotent of $e'Me'$. Then bm is idempotent in

$$e'Ae'=D\times e'Me'.$$

Hence $bm=be'$, since be' is primitive in A, and hence

$$be'\cdot A\cdot be'=e'Ae'$$

has the single idempotent be'. From Theorem 1, it follows that $m=e'$. Accordingly e' is a primitive idempotent of M.

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NOTES ON THE RATIONAL PLANE OSCNODAL QUARTIC CURVE*

BY J. H. NEELLEY

1. *Introduction.* In a recent paper†, which discussed the covariants of the curve R_2^4 with an oscnode, I made the statement that the quartic $\overline{12}^2(1, 2)$, where 1 and 2 refer to the base quartics $(\beta t)^4$ and $(\gamma t)^4$ of the fundamental involution, seemed to represent four collinear points of the curve when and only when the three nodes were adjacent in such a manner as to form an oscnode. In arriving at this conclusion it happened that, for each type of the curve examined, I chose as base quartics of the fundamental involution at least one catalectic quartic.‡ This accidental choice lead to the statement mentioned above. The failure of a later attempt to make the regular oscnodal invariants vanish under the condition that $\overline{12}^2(1, 2)$ should represent a line section of the curve caused the present investigation. In this paper, I shall discuss the form $\overline{12}^2(1, 2)$ and associated covariants and I shall develop several theorems which give new necessary and sufficient conditions for the oscnodal singularity.

2. *Certain Conditions.* I shall refer the curve to the tangents at the two points $t=0$ and $t=\infty$ and the line joining those points. This reference scheme assumes evidently that $t=0$ and $t=\infty$ are merely any two points of the curve such that the line of the curve at either does not pass through the other point. So this general reference triangle gives the curve as follows:

$$(1) \qquad \begin{cases} x_0 = a_0 t^4 + b_0 t^3 + c_0 t^2, \\ x_1 = \quad\; b_1 t^3 + c_1 t^2 + d_1 t, \\ x_2 = \quad\quad\; c_2 t^2 + d_2 t + e_2, \end{cases}$$

where neither a_0 nor e_2 can be zero, and in order to avoid some

* Presented to the Society, December 27, 1929.

† Neelley, *Effects of the oscnode upon covariant forms of the rational plane quartic curve*, this Bulletin, vol. 35, pp. 571–575.

‡ Neelley, *Compound singularities of the rational plane quartic curve*, American Journal of Mathematics, vol. 49 (1927), pp. 389–400.

singularity at $t=0$ or $t=\infty$ we shall say that neither b_1 nor d_1 shall vanish. This avoids at least a cusp at either point. Then we have

$$(2)\quad \begin{cases} (\beta t)^4 \equiv a_0b_1d_2t^4 + 4a_0b_1e_2t^3 - 4a_0d_1e_2t - b_0de_2, \\ (\gamma t)^4 \equiv a_0(c_1d_2 - c_2d_1)t^4 + 4a_0c_1e_2t^3 + 6a_0d_1e_2t^2 - c_0d_1e_2; \end{cases}$$

whence the form $\overline{12}^2(1, 2)$ except for the factor a_0e_2 is

$$(3)\quad \begin{cases} (a_0b_1d_1d_2 - 2a_0b_1c_1e_2)t^4 + 2(a_0c_2d_1^2 - a_0c_1d_1d_2 - a_0b_1d_1e_2)t^3 \\ \quad + (b_0c_2d_1^2 - b_1c_0d_1d_2 - b_0c_1d_1d_2 - 2a_0c_1d_1e_2)t^2 + 2(a_0d_1^2e_2 \\ \quad - b_1c_0d_1e_2 - b_0c_1d_1e_2)t - b_0d_1^2e_2. \end{cases}$$

This form and expressions (2) give as the conditions that $\overline{12}^2(1, 2)=0$ be a line section

$$(4)\quad B \equiv 2b_0b_1c_1e_2 - b_0b_1d_1d_2 - 2a_0b_1d_1e_2 - a_0c_1d_1d_2 + a_0c_2d_1^2 + b_1^2c_0e_2 = 0,$$

$$(5)\quad C \equiv 2b_1c_0c_1e_2 + b_0c_1^2e_2 - 2a_0c_1d_1e_2 - b_1c_0d_1d_2 - b_0c_1d_1d_2 + b_0c_2d_1^2 = 0,$$

where we have not written the non-vanishing factor $2a_0d_1e_2$ for each form.

Recalling that the catalectic sets of the fundamental pencil are associated with the nodes in such a way that each set fixes a node, let us consider the catalecticant of the form $\lambda(\beta t)^4+\mu(\gamma t)^4$. Except for the factor $a_0^2d_1e_2^2$, it may be written in the form

$$(6)\qquad A\lambda^3 + B\lambda^2\mu + C\lambda\mu^2 + D\mu^3 = 0,$$

where $A \equiv b_0b_1^2e_2 - a_0b_1d_1d_2$, B and C are forms (4) and (5) respectively, and $D \equiv c_0c_1^2e_2 - a_0d_1^2e_2 - c_0c_1d_1d_2 + c_0c_2d_1^2$. Consideration of conditions (4) and (5) shows that this catalecticant becomes

$$(7)\qquad A\lambda^3 + D\mu^3 = 0$$

when $\overline{12}^2(1, 2)=0$ is a line section. This gives the theorem:

If the form $\overline{12}^2(1, 2)$ of the fundamental pencil of the curve R_2^4 represents a line section, the catalecticant of the fundamental quartics reduces to the sum of two cubes.

The cubic (6) shows that the conditions (4) and (5) are not sufficient conditions to make nodes of the curve coincident.

But forms A and D are the conditions that the quartics (2) be catalectic respectively. So, if we pick as either $(\beta t)^4$ or $(\gamma t)^4$ one of the catalectic quartics of the fundamental pencil, the cubic (6) reduces to a perfect cube when $\overline{12}^2(1, 2)=0$ is a line section. Before stating the theorem this seems to give, let us consider the invariants which vanish when the curve has an oscnode.* With no loss of generality we assume $c_1=d_2=0$ in the representation (1) of the curve. Then the forms considered become

$$(8)\qquad \begin{cases} A \equiv b_0 b_1^2 e_2, & B \equiv a_0 c_2 d_1^2 - 2a_0 b_1 d_1 e_2 + b_1^2 c_0 e_2, \\ C \equiv b_0 c_2 d_1^2, & D \equiv c_0 c_2 d_1^2 - a_0 d_1^2 e_2, \end{cases}$$

except for factors involving a_0, d_1, and e_2. Obviously the tacnode invariant I_6' vanishes as the cubic has a triple root. So we need only observe the additional oscnodal invariant I_4. This becomes

$$(9)\qquad \begin{cases} 64a_0^3 d_1^3 e_2^3 [2A(27a_0 b_0 c_2 d_1^2 e_2 - 2b_0 c_0 c_2^2 d_1^2) + 2B(14a_0 b_1 c_0 c_2 d_1 e_2 \\ \quad - 3a_0^2 c_2 d_1^2 e_2 - 6a_0^2 b_1 d_1 e_2^2 - 3a_0 b_1^2 c_0 e_2^2 - a_0 c_0 c_2^2 d_1^2 \\ \quad - b_1^2 c_0^2 c_2 e_2) + C(a_0 b_0 c_2^2 d_1^2 - 10a_0 b_0 b_1 c_2 d_1 e_2)]. \end{cases}$$

Therefore $A=B=C=0$ are sufficient conditions for I_4 to vanish. Relations (8) show that A and C are not independent and $B=C=0$ are sufficient to make $I_4=0$ but they do not make $I_6'=0$. However, the independence of A and C is assured if we take only $c_1=0$ in the equations (1). In this case too we find that $A=B=C=0$ are sufficient to make I_4 vanish. We may now state these results as the following theorem.

THEOREM 1. *If the fundamental pencil of quartics of the curve be so chosen that one of the base quartics is catalectic, then the necessary and sufficient condition that the curve have an oscnode is that the covariant* $\overline{12}^2(1, 2)$ *represent a line section of the curve.*

The appearance of the forms B and C in the catalecticant naturally created a suspicion that they might occur in some of the forms of the system of the binary quartics (2). And it appears that if we operate with $(\beta t)^4$ upon the Hessian of $(\gamma t)^4$ and vice versa, the results are respectively

* Neelley, loc. cit., American Journal of Mathematics, vol. 49, p. 398.

(10) $\overline{12}^4(1,H_2) \equiv a_0^2 d_1 e_2^2 \cdot C, \qquad \overline{12}^4(2,H_1) \equiv a_0^2 d_1 e_2^2 \cdot B.$

These allow us to write the catalecticant cubic (6) in the following manner:

(11) $g_{3(1)}\lambda^3 + \overline{12}^4(2,H_1)\lambda^2\mu + \overline{12}^4(1,H_2)\lambda\mu^2 + g_{3(2)}\mu^3 = 0,$

showing that its coefficients are these invariants of the two quartics $(\beta t)^4$ and $(\gamma t)^4$. The forms (10) also give the following theorem.

THEOREM 2. *The form* $\overline{12}^2(1, 2)$ *is a line section of the curve when each base quartic of the fundamental pencil is apolar to the Hessian of the other. And when the proper choice of* $(\beta t)^4$ *or* $(\gamma t)^4$ *is made the curve has an oscnode if the base quartics are each apolar to the Hessian of the other and conversely.*

In testing this last theorem for the types of the curve with an oscnode we find that having picked for $(\beta t)^4$ a catalectic quartic the choice of $(\gamma t)^4$ is unrestricted and still each quartic is apolar to the Hessian of the other. But this is not true for any two quartics of the pencil. Then, on the assumption that $(\beta t)^4$ is catalectic, we find that any other member of the pencil such as $\lambda_1(\beta t)^4+\mu_1(\gamma t)^4$ is apolar to the Hessian of $(\beta t)^4$ and conversely if $\overline{12}^2(1, 2)$ represents a line section. The converse is readily shown when the general representation (1) of the curve is used.

In a previous article* we made the statement that the oscnodal cusp requires that each of the self-apolar members of the fundamental pencil be apolar to the Hessian of the other in addition to the invariant conditions. We find here that this is not the whole truth. For we have shown that this is true for the base quartics, if one is catalectic, for any type of the curve with an oscnode. This apparent inconsistency is at once removed, however, when we observe that if the oscnode is a cusp or a knot one of the self-apolar quartics is also catalectic. The situation is as follows.

THEOREM 3. *The curve with an oscnodal cusp has two distinct self-apolar members in its fundamental pencil, one of which is*

* Neelley, loc. cit., American Journal of Mathematics, vol. 49 (1927), p. 398.

catalectic, while the knot is such that the self-apolar members coincide. The knot then has one form representing both of its possible self-apolar forms and also its three coincident catalectic forms.

This shows that the previous statement is true but that it is a special case of the general situation, derived in this paper, for all types of the curve with an oscnode.

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THE TRANSFORMATIONS GENERATED BY AN INFINITESIMAL PROJECTIVE TRANSFORMATION IN FUNCTION SPACE*

BY I. A. BARNETT

The problem considered in this note is to determine the one-parameter group of finite transformations generated by an infinitesimal projective transformation in the space of continuous functions. It will be shown that this one-parameter group consists entirely of projective transformations of function-space in the sense defined by L. L. Dines.† While the problem is completely solved in the paper just cited (p. 57), it will be seen that the result may be obtained directly without the use of the auxiliary formulas developed by Dines in the first part of his paper.

Let the given infinitesimal projective transformation be defined by the integro-differential equation

$$(1)\qquad \frac{\partial\phi'(x;t)}{\partial t} = \lambda(x) + \mu(x)\phi'(x;t) + \int_0^1 \nu(x,y)\phi'(y;t)dy - \phi'(x;t)\int_0^1 \rho(y)\phi'(y;t)dy$$

with the boundary condition $\phi'(x;0)=\phi(x)$. It is required to

* Presented to the Society, April 6, 1928.

† *Projective transformations in function space*, Transactions of this Society, vol. 20 (1914), p. 45.

prove that the unique solution of equation (1), with the given boundary condition, has the form

$$(2)\quad \phi'(x;t) = \frac{\alpha(x;t) + \beta(x;t)\phi(x) + \int_0^1 \gamma(x,y;t)\phi(y)dy}{\delta(t) + \int_0^1 \epsilon(y;t)\phi(y)dy},$$

where the functions α, β, γ, δ and ϵ are to be determined in terms of λ, μ, ν, and ρ.

First introduce new variables by means of the transformation*

$$(3)\qquad \Phi(x;t) = \omega(t)\phi(x;t), \quad \Phi'(x;t) = \omega'(t)\phi'(x;t).$$

For $t=0$, these become

$$(4)\qquad \Phi(x) = \omega\phi(x), \qquad \Phi'(x) = \omega'\phi'(x),$$

where

$$\Phi(x) = \Phi(x;0), \qquad \omega = \omega(0).$$

One readily finds by means of (3) that the single equation (1) may be replaced by the pair of equations

$$(1')\begin{cases} \dfrac{\partial\Phi'(x;t)}{\partial t} = \lambda(x)\omega'(t) + \mu(x)\Phi'(x;t) + \displaystyle\int_0^1 \nu(x,y)\Phi'(y;t)dy, \\ \dfrac{\partial\omega'(t)}{\partial t} = \displaystyle\int_0^1 \rho(y)\Phi'(y;t)dy, \end{cases}$$

with the boundary conditions $\Phi'(x;0)=\Phi(x)$, $\omega'(0)=\omega$. Likewise, equation (2) goes into the system

* Kowalewski, *Sur une classe de transformations infinitésimales*, Comptes Rendus, vol. 153 (1911), p. 1452. Kowalewski makes no specific mention of the use of homogeneous coordinates in function space, but the method he gives for reducing the form of an infinitesimal projective transformation to one involving only linear operations is precisely this change of coordinates. In a paper published about eleven years later, Dines actually points out how, by the introduction of such coordinates, he is able to simplify, to a considerable extent, some of the results in his Transactions paper cited above. In this last paper, which is entitled *A primary classification of projective transformation in function space* (American Journal of Mathematics, vol. 44 (1925)), he suggests the possibility of recasting other results in his Transactions paper. The present note does this for §9 of his paper.

$$(2')\quad \begin{cases} r\Phi'(x;t) = \alpha(x;t)\omega + \beta(x;t)\Phi(x) + \int_0^1 \gamma(x,y;t)\Phi(y)dy, \\ r\omega'(t) = \int_0^1 \epsilon(y;t)\Phi(y)dy + \delta(t)\omega, \end{cases}$$

where r is an arbitrary constant.

The result of differentiating (2′) with respect to t is

$$(5)\quad \begin{cases} r\dfrac{\partial\Phi'(x;t)}{\partial t} = \dfrac{\partial\alpha(x;t)}{\partial t}\omega + \dfrac{\partial\beta(x;t)}{\partial t}\Phi(x) \\ \qquad\qquad + \displaystyle\int_0^1 \frac{\partial\gamma(x,y;t)}{\partial t}\Phi(y)dy, \\ r\dfrac{\partial\omega'(t)}{\partial t} = \displaystyle\int_0^1 \frac{\partial\epsilon(y;t)}{\partial t}\Phi(y)dy + \frac{\partial\delta(\epsilon)}{\partial t}\omega. \end{cases}$$

Substituting (1′) in (5), one finds the relations

$$(6)\quad \begin{cases} r\left[\lambda(x)\omega'(t) + \mu(x)\Phi'(x;t) + \displaystyle\int_0^1 \nu(x,y)\Phi'(y)dy\right] \\ \qquad = \omega\dfrac{\partial\alpha(x;t)}{\partial t} + \Phi\dfrac{\partial\beta(x;t)}{\partial t} + \displaystyle\int_0^1 \frac{\partial\gamma(x,y;t)}{\partial t}\Phi(y)dy, \\ r\left[\displaystyle\int_0^1 \rho(y)\Phi'(y;b)dy\right] = \displaystyle\int_0^1 \frac{\partial\epsilon(y;t)}{\partial t}\Phi(y)dy + \omega\frac{\partial\delta(t)}{\partial t}. \end{cases}$$

Replacing $\Phi'(x;t)$ and $\omega'(t)$ in the left member of (6) by their expressions from (2′), the following equations for the determination of $\alpha, \beta, \gamma, \delta, \epsilon$ are obtained:

$$(7)\quad \begin{cases} \omega[\lambda(x)\delta(t) + \mu(x)\alpha(x;t)] + \Phi(x)[\mu(x)\beta(x;t)] \\ \quad + \displaystyle\int_0^1 \Phi(y)[\lambda(x)\epsilon(y;t) + \nu(x,y)\beta(y;t) + \mu(x)\gamma(x,y;t) \\ \quad + \displaystyle\int_0^1 \nu(x,z)\gamma(z,y;t)dz]dy = \omega\frac{\partial\alpha(x;t)}{\partial t} + \Phi(x)\frac{\partial\beta(x;t)}{\partial t} \\ \quad + \displaystyle\int_0^1 \frac{\partial\epsilon(y;t)}{\partial t}\Phi(y)dy, \\ \omega\displaystyle\int_0^1 \rho(y)\alpha(y;t)dy + \int_0^1 \Phi(x)[\rho(x)\beta(x;t) \\ \quad + \displaystyle\int_0^1 \rho(x)\gamma(x,y;t)dy]dx = \omega\frac{\partial\delta(t)}{\partial t} + \int_0^1 \frac{\partial\epsilon(x;t)}{\partial t}\Phi(x)dx. \end{cases}$$

Now it is easy to see that if

$$\omega A(x) + B(x)\Phi(x) + \int_0^1 C(x,y)\Phi(y)dy$$

vanishes identically for all real functions $\Phi(x)$ and all real numbers ω, then $A(x) \equiv B(x) \equiv C(x, y) \equiv 0$. Applying this result to equations (7), one finds that the required coefficients of the projective transformations satisfy the following five linear homogeneous integro-differential equations:

$$\frac{\partial\alpha(x;t)}{\partial t} = \mu(x)\alpha(x;t) + \lambda(x)\delta(t) + \int_0^1 \nu(x,y)\alpha(y;t)dy,$$

$$\frac{\partial\beta(x;t)}{\partial t} = \mu(x)\beta(x;t),$$

$$\frac{\partial\gamma(x,y;t)}{\partial t} = \lambda(x)\epsilon(y;t) + \nu(x,y)\beta(y;t) + \mu(x)\gamma(x,y;t) + \int_0^1 \nu(x,z)\gamma(z,y;t)dz,$$

$$\frac{\partial\delta(t)}{\partial t} = \int_0^1 \rho(y)\alpha(y;t)dy,$$

$$\frac{\partial\epsilon(x;t)}{\partial t} = \rho(x)\beta(x;t) + \int_0^1 \rho(y)\gamma(y,x;t)dy.$$

The initial conditions are

$$\alpha(x;0) \equiv \gamma(x,y;0) \equiv \epsilon(x;0) \equiv 0; \quad \beta(x;0) \equiv \delta(0) \equiv 1.$$

These are the equations (19) obtained by Dines in the paper already cited, and, as indicated there, their solution may be found as power series in t.

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INVERSE CORRESPONDENCES IN AUTOMORPHISMS OF ABELIAN GROUPS*

BY G. A. MILLER

The totality of the operators which correspond to their inverses in an automorphism of an abelian group G obviously constitutes a subgroup of G. It is well known that G cannot contain any characteristic operator besides the identity unless this operator is of order 2, and that it cannot contain more than one characteristic operator of this order. Hence it results that whenever G contains any characteristic operator besides the identity this operator must appear in every subgroup which is composed of all the operators of G which correspond to their inverses in a given automorphism of G. We proceed to prove that a necessary and sufficient condition that an automorphism of G can be established such that all the operators of a given subgroup H correspond to their inverses, while no other operator of G satisfies this condition, is that H involves all the characteristic operators of G. In particular, when G involves no characteristic operator besides the identity, it is possible to establish an automorphism of G such that all the operators of an arbitrary subgroup correspond to their inverses, while no other operator of G has this property.

When G is of order p^m, p being a prime number, and of type $(1, 1, 1, \cdots)$ it is well known that all of its operators besides the identity can be transformed cyclically and that none of its subgroups of order p is transformed into itself under such a transformation whenever $m>1$. Moreover, when all the operators of G are arranged in co-sets with respect to H, where each of the operators of H corresponds to its inverse, then either all the operators of a co-set correspond to their inverses in the same automorphism or none of these operators has this property. Hence the theorem under consideration is obviously true when G is of order p^m and of type $(1, 1, 1, \cdots)$. Moreover, it results from the fact that an abelian group contains only one

* Presented to the Society, December 30, 1929.

Sylow subgroup of a given order that we may confine our attention to the case when the order of G is p^m in the proof of this theorem. We therefore assume that this condition is satisfied and that H is any subgroup of G which involves all the characteristic operators of G. If all the operators of G which are not in H are included in co-sets of G with respect to H, it is clear that these co-sets can be so ordered that each of the first k_1 involves a power of one independent generator of G, and none of the remaining ones has this property, while each of those of the next k_2 involves either a power of a second independent generator or the product of such a power and a power of the first independent generator, and none of the remaining ones has this property, etc.

We may now suppose that an automorphism of G has been established in which all the operators of H correspond to their inverses. When $p>2$ it is clear that this automorphism can be extended so that no power of the first independent generator corresponds to its inverse except its powers which are in H. Similarly, no power of the second independent generator except its powers contained in H needs to correspond to its inverse in such an automorphism. As this process may be continued until the operators of G have been exhausted, it results that no operator of G except those of H corresponds to its inverse in the resulting automorphism. When $p=2$ and G involves only one independent generator of highest order, H must involve the operator of order 2 which is a power of this generator, and we may evidently proceed as before except that the following independent generators are replaced by their products into this one when this is used as the first independent generator in the given process and the operator of order 2 which is a power of such an independent generator is not in the earlier co-sets. When this operator is in an earlier co-set no modification is necessary. Hence, we have established the following theorem.

THEOREM 1. *A necessary and sufficient condition that the operators of a given subgroup of an abelian group may correspond to their inverses in an automorphism of this group in which no other operator corresponds to its inverse is that this subgroup involves all the characteristic operators of the group.*

When G is a cyclic group of order p^m, $m>1$, and all the

operators of its subgroup of order p^α correspond to their inverses, while no other operator has this property, it is very easy to see that the order of this automorphism is $2p$ whenever $\alpha=m-1$ and p is odd, and that there are $p-1$ such automorphisms of G, since each of the remaining operators can correspond to its inverse multiplied by an operator of order p contained in G in such an automorphism. When $p=2$ there is only one such automorphism and its order is 2. The latter corresponds to the case when every operator of G is transformed into its $2^{m-1}-1$ power. In general, when p is odd the automorphisms in which exactly p^α operators correspond to their inverses, $\alpha>0$, are of order $2p^{m-\alpha}$, since in the square of such an automorphism exactly p^α operators correspond to themselves. When $p=2$ and $1<\alpha<m-1$ these automorphisms are of order $2^{m-\alpha}$, since in the square of such an automorphism exactly $2^{\alpha+1}$ of the operators correspond to themselves. Hence we have established the following theorem.

THEOREM 2. *If exactly p^α, $\alpha>0$, of the operators of a cyclic group of order p^m correspond to their inverses in an automorphism of this group, then this automorphism is of order $2p^{m-\alpha}$ whenever p is an odd prime number and also when $p=2$ and $m=\alpha$. When $p=2$ and $1<\alpha<m$ this automorphism is of order $2^{m-\alpha}$, while it may have any order from 1 to 2^{m-2} when $\alpha=1$.*

By means of this theorem it is easy to determine the order, or the possible orders, of the automorphisms of any cyclic group G in which the operators of a given subgroup H correspond to their inverses. If the order of H is divisible by all the odd prime factors of the order of G, then the order of each of these automorphisms is equal to a power of 2 into the largest odd factor of the index of H under G. When the order of G is odd this power of 2 is 2, and when the order of H is divisible by 4 but not by the highest power of 2 which divides the order of G, then this power of 2 is equal to the highest power of 2 which divides the index of H under G. When the order of H is not divisible by some prime factor which divides the order of G, then the order of the corresponding automorphisms may be the product of the order of any operator in the group of isomorphisms of the Sylow subgroup of G whose order is a power of this prime factor and the order as determined above. A

necessary and sufficient condition that there is only one automorphism in which all the operators of a proper subgroup of G correspond to their inverses, while no other operator of G has this property, is that this subgroup is of index 2 and involves an operator of order 2.

In order to determine the number of the automorphisms whose orders were determined above it may first be noted that when the order of G is of the form p^m and the order of H is p^α, $\alpha>0$, the number of the possible automorphisms is $p^{m-\alpha}-p^{m-\alpha-1}$, since a generator of G must correspond to its inverse multiplied by an operator of order $p^{m-\alpha}$ in such an automorphism and all such products will obviously give rise to such an automorphism. This results also from the following theorem.

THEOREM 3. *A necessary and sufficient condition that an automorphism of the cyclic group of order p^m, p being an odd prime number, is of order $2p^\beta$ is that $p^{m-\beta}$ of its operators correspond to their inverses under this automorphism.*

A proof of this theorem results almost directly from the fact that the square of such an automorphism is of order p^β and hence leaves invariant $p^{m-\beta}$ of the operators of G. The corresponding theorem when $p=2$ is somewhat less elegant since we have to restrict it by saying that when the operators of order 4 in an automorphism of order 2^α of a cyclic group of order 2^m correspond to their inverses then exactly $2^{m-\alpha}$ of the operators of this cyclic group must correspond to their inverses in this automorphism whenever $\alpha>1$. When $\alpha=1$ this number is either 2^{m-1} or 2^m.

Since all the commutators which arise from an automorphism of order 2 of any group whatever correspond to their inverses under this automorphism, it results that in every automorphism of order 2 of an abelian group the number of the operators which correspond to their inverses is a divisor of the product of the order of the group K generated by the operators of order 2 which correspond to themselves under this automorphism and the order of the corresponding commutator subgroup. It is also a multiple of the order of K and the order of the quotient group of this commutator subgroup with respect to its cross-cut with the subgroup composed of the operators which correspond

to themselves under this automorphism. Moreover, it is possible to construct a group of order 2^m, where m is any given number, which admits an automorphism of order 2 in which the number of inverse correspondences is an arbitrary number which satisfies these two conditions and the order of K is any divisor of 2^m. In particular, *the number of inverse correspondences under an automorphism of order* 2 *of any abelian group of odd order is always equal to the order of the corresponding commutator subgroup.* In fact, in an automorphism of order 2 of any group whatsoever of odd order a necessary and sufficient condition that an operator corresponds to its inverse is that it is a commutator under this automorphism, but these commutators do not necessarily constitute a group when the given group of odd order is non-abelian.

THE UNIVERSITY OF ILLINOIS

THE FOURTH POSTULATES OF RIESZ AND HAUSDORFF*

BY ORRIN FRINK JR.

A question which has remained unanswered for some time is whether every space which satisfies the four postulates of Riesz for points of accumulation and the additional postulate that every derived set is closed, is necessarily a Hausdorff space. A paper on this subject was recently presented to the American Mathematical Society.† I have succeeded in constructing an example, given below, which shows that the answer is in the negative.

A good discussion of the two postulate systems of Riesz and Hausdorff may be found in Fréchet's recent Borel monograph, *Les Espaces Abstraits*. Although Hausdorff's postulates are in terms of neighborhoods and those of Riesz concern the accumulation points directly, there is almost a complete parallelism between them, provided we add to Riesz's four, as Fréchet does, a fifth postulate that every derived set is closed. The only

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* Presented to the Society, December 27, 1929.

† R. G. Putnam, this Bulletin, vol. 35 (1929), p. 442.

uncertainty is caused by the fact that Hausdorff's fourth postulate is apparently slightly stronger than Riesz's number four. The postulates of Riesz are as follows:

1. Every point of accumulation of a set E is a point of accumulation of every set which includes E as a subset.

2. Every point of accumulation of the sum of two sets is a point of accumulation of at least one of the two sets.

3. A set consisting of a single point has no points of accumulation.

4. Every point of accumulation of a set E is determined by the knowledge of all the subsets of E of which this point is a point of accumulation.

Postulate 4 is interpreted as meaning that if a and b are two distinct points of accumulation of a set E, there exists at least one subset of E of which one of the points a and b, and one alone, is a point of accumulation.

The fourth postulate of Hausdorff requires that if x and y are any two distinct points, there exist neighborhoods U_x and V_y of x and y respectively, which have no points in common.

The example is a space consisting of:

I. All points of the XY plane with positive x and y coordinates. A neighborhood of such a point is defined as usual to consist of the points inside a circle with the point as center and with any radius small enough so that only points with positive x and y coordinates are included.

II. The point O (the origin). A neighborhood of O is defined to be a quarter circle consisting of O and all points with positive x and y coordinates lying inside a circle with O as center.

III. The point P. This point is given abstractly and may be thought of as replacing all positive points of the x-axis. A neighborhood of P by definition is to consist of P and all points lying beneath some curve $y=f(x)$, where $f(x)$ is a continuous, single valued, everywhere positive function defined for all positive values of x. The points O and P are not contained in any neighborhoods but their own.

From this definition it follows that O is an accumulation point of a set when and only when the set contains a sequence of points converging in the ordinary sense to O, and P is an accumulation point of a set when and only when the set contains a sequence of points converging in the ordinary sense to some

positive point of the X-axis (which isn't there, of course, in our space). A sequence of points converging to O, however, cannot have P for an accumulation point. For we can select our curve $y=f(x)$ in such a way that all points of the sequence other than P lie above it.

This space satisfies all the postulates of Riesz when accumulation points have been defined as usual in terms of neighborhoods. Postulate 1 is satisfied in every neighborhood space. Postulate 2 follows from the fact that given any two neighborhoods of a point there is a third neighborhood contained in them both. Postulate 3 follows from the fact that given any two points there is a neighborhood of one of them that does not contain the other, and conversely. Riesz's postulate 4 is also satisfied. This requires that we be able to distinguish between two accumulation points of a set by finding a subset of which one is an accumulation point and the other not. If neither of these two points is O or P, we obviously have this property, since our space is cartesian if we omit O and P. In fact, any set of which a given point is an accumulation point has a subset of which this point is the unique accumulation point. But even if one of the points is O or P, we have seen above that if either of these two points is an accumulation point of a set M, it is the unique accumulation point of a sequence of points picked out of M. The fifth postulate, that every derived set is closed, follows from the fact that our neighborhoods are open sets.

However, Hausdorff's fourth postulate is not satisfied. For every neighborhood of O has points in common with every neighborhood of P. Furthermore, this will be true if we replace our definition of neighborhoods by any equivalent definition (equivalent in the sense of leading to the same definition of accumulation points). For every new neighborhood would have to contain an old neighborhood as a subset. The essential feature of this example is then that although all neighborhoods of O and P have points in common, we can always distinguish between O and P as accumulation points of a set by finding a subset of which one is an accumulation point but not the other.

PENNSYLVANIA STATE COLLEGE

ON THE REPRESENTATION OF ANALYTIC FUNCTIONS OF SEVERAL VARIABLES AS INFINITE PRODUCTS*

BY J. M. FELD AND P. NEWMAN

1. *Introduction.* In a paper by J. F. Ritt soon to appear in the Mathematische Zeitschrift, he proves that any function $f(z)$, analytic, and equal to unity at $z=0$, can be represented in one and only one way as an infinite product $\prod_1^\infty(1+c_nz^n)$, which converges absolutely for $|z|\leqq 1/(6R)$, where R is the least upper bound of the infinite sequence $|b_1|$, $|b_2|^{1/2}$, $\cdots$, $|b_k|^{1/k}$, $\cdots$ and b_k is the coefficient of z^{k-1} in the Taylor expansion of $f'(z)/f(z)$.

The object of this paper is to extend this result to functions of two variables. The first part will be concerned with a demonstration that an analytic function $f(x, y)=1+\sum b_{mn}x^my^n$ can be uniquely represented as an absolutely convergent infinite product $\prod(1+a_{mn}x^my^n)$ with constant a's. The second part will consider the representation of $f(x, y)$ in the form $\prod_1^\infty(1+P_n)$ where P_n is a homogeneous polynomial in x and y of degree n. Although the proof in each case is carried out for two variables, it will be evident how to extend it to functions of any number of variables. It should be noted, however, that the analytic functions considered in this paper constitute a restricted class of such functions, namely, those which equal unity at the origin. For one variable the corresponding assumption is not an essentially restrictive one.

2. THEOREM 1. *If $f(x, y)$ is analytic and equal to unity at $(0, 0)$, then in the neighborhood of $(0, 0)$ it admits a unique representation as an absolutely convergent infinite product*

$$f(x,y) = \prod_{m,n}(1 + a_{mn}x^my^n)$$

with constant a_{mn}.

Let the Taylor's expansion of $f(x, y)$ at $(0, 0)$ be $1+\sum b_{mn}x^my^n$. Since $f(0, 0)=1$ there is a neighborhood about $(0, 0)$ for which

* Presented to the Society, October 26, 1929.

$\log f(x, y)$ will be represented by an absolutely convergent expansion $\sum C_{mn}x^m y^n$. By Lemaire's generalization of a theorem of Cauchy (Bulletin des Sciences Mathématiques, 1896, p. 286), the set $|C_{mn}|^{1/(m+n)}$ is bounded; we denote the least upper bound by r.

Assuming the product expansion and taking logarithms, we find

$$\sum_{m,n} C_{mn}x^m y^n = \sum_{m,n,p}(-1)^{p+1}\frac{(a_{mn}x^m y^n)^p}{p}. \tag{1}$$

We denote the common divisors of a fixed pair of m and n by $d_0=1, d_1, \cdots, d_t=D$, where D is the greatest common divisor. Observing that

$$\frac{(a_{m/d_i,n/d_i}x^{m/d_i}y^{n/d_i})^{d_i}}{d_i} = \frac{(a_{m/d_i,n/d_i})^{d_i}x^m y^n}{d_i},$$

we find for the coefficient of $x^m y^n$ in the second member of (1),

$$\sum_{i=0}^{t}(-1)^{d_i+1}\frac{1}{d_i}(a_{m/d_i,n/d_i})^{d_i}.$$

Equating coefficients of $x^m y^n$, we have

$$C_{mn} = a_{mn} + \sum_{1}^{t} \pm \frac{1}{d_i}(a_{m/d_i,n/d_i})^{d_i}. \tag{2}$$

If neither m nor n is zero, then (2) will not involve any a with a zero subscript. This is also evident from the consideration that a zero subscript represents a term in the product of the form $(1+a_{0n}y^n)$ which could not give rise to a term containing x when expanded in a logarithmic series. It follows from our hypothesis that the subscripts 0 0 never occur.

When m or n is unity, (2) reduces to

$$C_{1n} = a_{1n}. \tag{3}$$

From (2) we get

$$|a_{mn}| \leqq |C_{mn}| + \sum_{1}^{t}\frac{1}{d_i}|a_{m/d_i,n/d_i}|^{d_i}. \tag{4}$$

If $(1/d_k)|a_{m/d_k,n/d_k}|^{d_k}$ is the greatest of the terms following C_{mn} in (4), then since the number of common divisors cannot exceed $D/2$, we obtain

$$|a_{mn}| \leqq |C_{mn} + \frac{D}{2}| a_{m/d_k,n/d_k}|^{d_k}. \tag{5}$$

There are two possibilities to be considered in (5); (A) if the second term on the right of (4) is less than or equal to the first, then $|a_{mn}|^{1/(m+n)} \leqq (2|C_{mn}|)^{1/(m+n)} \leqq 2r$, otherwise, (B)

$$|a_{mn}| < D|a_{m/d_k,n/d_k}|^{d_k}.$$

Since $D^{1/(m+n)} \leqq D^{1/D}$, we find from (B)

$$|a_{mn}|^{1/(m+n)} < D^{1/D}|a_{m/d_k,n/d_k}|^{d_k/(m+n)}. \tag{6}$$

We can repeat the reasoning of (A) and (B) employing

$$|a_{m/d_k,n/d_k}|^{d_k/(m+n)} = |a_{m/d_k,n/d_k}|^{1/(m/d_k+n/d_k)}.$$

If (B) holds again, the inequality of the type (6) will now involve the highest common divisor of m/d_k, n/d_k, say D_1. Since $d_k \neq 1$, $D \geqq 2D_1$. Repeating the process until we come to an "a" for which (A) holds, we have from (A) and (6)

$$|a_{mn}|^{1/(m+n)} < D^{1/D}D_1^{1/D_1} \cdots D_l^{1/D_l}(2r). \tag{7}$$

The D's are a decreasing sequence, $D_{j+1} \leqq 2D_j$, which at the worst will end at $D_l = 2$. This is seen readily from (3), (5), and condition (A).

From (7), we find

$$\log|a_{mn}|^{1/(m+n)} < \frac{1}{D}\log D + \frac{1}{D_1}\log D_1 + \cdots + \log 2r.$$

We consider first the case $D_l \neq 3$, and since $D_{j+1} \leqq 2D_j$, we have

$$\log|a_{mn}|^{1/(m+n)} < \log 2r + \sum_1^\infty \frac{1}{2^k}\log 2^k = \log 2r + \log 4 = \log 8r.$$

When $D_l = 3$,

$$\log|a_{mn}|^{1/(m+n)} < \log 2r + \sum_0^\infty \frac{1}{3\cdot 2^k}\log(3\cdot 2^k)$$

$$= \log 2r + \frac{1}{3}(\log 3)\sum_0^\infty \frac{1}{2^k} + \frac{1}{3}\sum_1^\infty \frac{\log 2^k}{2^k}$$

$$= \log 2r + \frac{2}{3}\log 3 + \frac{1}{3}\log 4 < \log 8r.$$

In any case,

$$|a_{mn}|^{1/(m+n)} < 8r.$$

By Lemaire's theorem, the associated power series $\sum a_{mn}x^m y^n$ is absolutely convergent for the associated regions whose radii are given by $|\rho||\rho'|<1/(8r)$ and therefore in any such pair of regions $f(x, y)$ will be represented by the absolutely convergent product $\prod(1+a_{mn}x^m y^n)$. From the form of the recursion formula it follows at once that the representation is unique. The theorem can be extended to any number of variables by a similar proof.

3. THEOREM 2. *If $f(x, y)$ is analytic in the neighborhood of the origin and if $f(0, 0)=1$, there exists a region $|x| \leqq \rho, |y| \leqq \rho$ in which $f(x, y)$ can be represented in one and only one way as an absolutely convergent infinite product $\prod_1^\infty(1+Q_n)$, where Q_n is a homogeneous polynomial in x and y of degree n.*

Let the Taylor expansion of $f(x, y)$ at $(0, 0)$ be written in the form

$$f(x,y) = 1 + P_1 + P_2 + \cdots + P_n + \cdots, \tag{8}$$

where P_n is a homogeneous polynomial in x and y of degree n. Then there exists a ρ_1 such that (1) converges absolutely for $|x|<\rho_1$ and $|y|<\rho_1$. The function of x, y and t, $f(xt, yt)$ is analytic for x, y and t small and has the expansion

$$f(xt, yt) = 1 + \sum_{n=1}^{\infty} P_n t^n.$$

For small values of x, y and t, $\log f(xt, yt)$ is analytic and has an expansion

$$\log f(xt,yt) = A_1 t + A_2 t^2 + \cdots + A_n t^n + \cdots, \tag{9}$$

where A_n is a homogeneous polynomial in x and y of degree n, and where the set of functions $|A_n|^{1/n}$ has an upper bound. Let r be the least upper bound of this set in the region in which (9) holds.

Setting

$$1 + \sum_1^\infty P_n t^n = \prod_1^\infty (1 + Q_n t^n), \tag{10}$$

we obtain $P_1=Q_1$, $P_2=Q_2$, $P_3=Q_3+Q_1Q_2$, etc., and it is evident

that we can solve for all the Q's by means of recursion formulas, and that Q_n is homogeneous in x and y of degree n. Taking logarithms of both members of (10) and equating like powers of t, we get

$$A_n = Q_n \pm \frac{1}{d_1} Q_{n/d_1}^{d_1} \pm \frac{1}{d_2} Q_{n/d_2}^{d_2} \pm \cdots \pm \frac{1}{n} Q^n, \tag{11}$$

where d_i is an integral divisor of n. Therefore

$$|Q_n| \leqq |A_n| + \frac{n}{2} \left| \frac{1}{D_1} Q_{n/D_1}^{D_1} \right|,$$

where

$$\left| \frac{1}{D_1} Q_{n/D_1}^{D_1} \right|$$

is the greatest of the terms

$$\left| \frac{1}{d_i} Q_{n/d_i}^{d_i} \right|, \quad (d_i > 1).$$

By a procedure like that used in the proof of Theorem 1, we arrive at the inequality

$$|Q_n|^{1/n} < n^{1/n} D^{1/D_1} D^{1/D_2} \cdots D^{1/D_i}(2r).$$

We therefore have, as in the preceding case, $|Q_n|^{1/n} < 8r$. Thus there exists a ρ_2 such that $\sum |Q_n| t^n$ converges for $|x| \leqq \rho_2$, $|y| \leqq \rho_2$, $|t| \leqq 1/(8r)$. Therefore $\prod(1+Q_n t^n)$ must converge absolutely in this region. Since

$$f(xt, yt) = \prod [1 + Q_m(xt, yt)],$$

we have the result that

$$f(x, y) = \prod(1 + Q_n)$$

in the region $|x| \leqq \rho = \rho_2/(8r)$, $|y| \leqq \rho = \rho_2/(8r)$.

COLUMBIA UNIVERSITY

ON THE NATURE OF θ IN THE MEAN-VALUE THEOREM OF THE DIFFERENTIAL CALCULUS*

BY GANESH PRASAD

1. *Introduction.* If $f(x)$ is a single-valued function which is finite and continuous in an interval (a, b), the ends being included, than the relation

$$(\mathrm{M}) \qquad f(x+h) = f(x) + hf'(x+\theta h), \qquad 0 < \theta < 1,$$

holds for every value of x and h for which the interval $(x, x+h)$ is in the interval (a, b); provided that *either* $f'(x)$ exists at every point inside the interval (a, b) *or* a certain less restrictive condition† is satisfied. In recent years the nature of θ has been studied by a number of writers‡ who start with the assumption that $f''(x)$ exists everywhere in the interval (a, b). The two theorems, which it is the object of this paper to formulate and prove, are believed to be new and hold even if $f''(x)$ does not exist everywhere. For the sake of clarity and fixity of ideas, I consider θ only as a function of h, assuming x to be a constant, say 0, in the theorem (M).

2. THEOREM I. *If $\theta(h)$ is single-valued and continuous, it is not necessarily differentiable for every value of h.*

PROOF. Take $f(x)$ to be the indefinite integral of a monotone, increasing and continuous function which has a differential coefficient everywhere in the interval (a, b), excepting the points

* Presented to the Society, February 25, 1928.

† The condition of W. H. Young and G. C. Young, Quarterly Journal of Mathematics, vol. 40 (1909), p. 1; Hobson's *Theory of Functions of a Real Variable*, vol. 1, 3d edition, 1927, p. 384; or the still less restrictive condition of A. N. Singh, Bulletin of the Calcutta Mathematical Society, vol. 19 (1928), p. 43.

‡ R. Rothe (Mathematische Zeitschrift, vol. 9 (1921), p. 300; Tôhoku Mathematical Journal, vol. 29 (1928), p. 145); T. Hayashi (Science Reports of the Tôhoku Imperial University, (1), vol. 13 (1925), p. 385); O. Szász (Mathematische Zeitschrift, vol. 25 (1926), p. 116).

of an everywhere dense set. Such a function is that given by T. Broden.* Denoting Broden's function by w, let

$$f(x) = \int_0^x w(t)dt,$$

and let the everywhere dense set be denoted by S; also let ξ stand for $h\theta$. Then it is easily seen that ξ is a single-valued and continuous function of h, and that, corresponding to each value of ξ, there is a value of h and only one value. Now (M) gives

$$f(h) = hf'(\xi) = hw(\xi),$$

whatever h may be.

Therefore, as $f'(h)$ exists,

$$\frac{d}{dh}\{hw(\xi)\}, \quad \text{that is,} \quad w(\xi) + h\frac{dw}{dh},$$

must exist for every value of h. Thus, at any point $h=h'$ which corresponds to a point $\xi=\xi'$ of S, $d\xi/dh$ and, consequently, $d\theta/dh$ must be non-existent; otherwise $w'(\xi')$ will exist which is impossible.

Therefore it is proved that, for every value of h corresponding to which ξ is a point of S, $d\theta/dh$ is non-existent.

3. THEOREM II. *If $\theta(h)$ is single-valued, it is necessarily continuous for every value of h.*

PROOF. Assume, if possible, that $\bar{h}$ is a point of discontinuity of $\theta(h)$. Then, denoting the corresponding values of ξ and θ by $\bar{\xi}$ and $\bar{\theta}$ respectively, we have by (M)

$$f(\bar{h}) = \bar{h}f'(\bar{\xi}).$$

Now two possibilities arise: the discontinuity may be of the first kind or of the second kind.

(a) If the discontinuity is of the first kind, then there must be a sequence $\{h_n\}$, tending to $\bar{h}$, for which the corresponding sequence $\{\xi_n\}$ does not tend to $\bar{\xi}$ but to $\bar{\xi}'$ different from $\bar{\xi}$. Thus

$$f'(\bar{\xi}) = f'(\bar{\xi}').$$

* Journal für Mathematik, vol. 118, p. 27; Hobson's *Theory of Functions of a Real Variable*, vol. 1, 1927, p. 389.

So, for the same value of h, namely, $\bar{h}$, there are two values of θ, namely, $\bar{\theta}$ and $\bar{\theta}'$, which is absurd, since θ is single-valued.

(b) If the discontinuity is of the second kind, then there must be a sequence $\{h_n\}$, tending to $\bar{h}$, for which the corresponding sequence $\{\xi_n\}$ does not tend to any limit. Therefore two values k_1 and k_2 of h can always be found as near as we please to $\bar{h}$ such that the corresponding values η_1 and η_2 of ξ differ from each other by a quantity greater than a suitably prescribed positive quantity δ. But, from (M), $f(h)/h$ and, consequently, $f'(\xi)$ are continuous functions of h at $\bar{h}$. Therefore ξ must be multiple-valued at $\bar{h}$, which is absurd, since θ is single-valued.

THE UNIVERSITY OF CALCUTTA

A NUMERICAL FUNCTION APPLIED TO CYCLOTOMY

BY EMMA T. LEHMER

A function $\phi_2(n)$ giving the number of pairs of consecutive integers each less than n and prime to n, was considered first by Schemmel.* In applying this function to the enumeration of magic squares, D. N. Lehmer† has shown that if one replaces consecutive pairs by pairs of integers having a fixed difference λ prime to $n = \prod_{i=1}^{t} p_i^{\alpha_i}$, then the number of such pairs (mod n) whose elements are both prime to n is also given by

$$\phi_2(n) = \prod_{i=1}^{t} p_i^{\alpha_i - 1}(p_i - 2) .$$

As is the case for Euler's totient function $\phi(n)$, the function $\phi_2(n)$ obviously enjoys the multiplicative property $\phi_2(m)\phi_2(n) = \phi_2(mn)$, $(m, n) = 1$, $\phi_2(1) = 1$. *In what follows we call an integer simple if it contains no square factor* > 1. For a simple number n we have the following analog of Gauss' theorem:

$$\sum_{\delta \mid n} \phi_2(\delta) = \phi(n), \tag{1}$$

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* Journal für Mathematik, vol. 70 (1869), pp. 191–2.

† Transactions of this Society, vol. 31 (1929), pp. 538–9.

where n is simple and where the summation extends over all the divisors of n.* Using Dedekind's inversion formula, we can write

$$\sum_{\delta|n} \phi(n/\delta)\mu(\delta) = \phi_2(n), \tag{2}$$

where n is simple and where $\mu(n)$ is Merten's inversion function.*

It is the purpose of this note to develop another property of $\phi_2(n)$, true only for simple numbers, and apply it to the evaluation of the discriminants and resultants of cyclotomic equations.

Let $\{\lambda_k\}$ denote the set of numbers less than n and prime to n, and let λ be any number of this set. Then it follows at once from D. N. Lehmer's result, that there are $\phi_2(n)$ numbers prime to n in the set $\{\lambda+\lambda_k\}$. Let us inquire how many numbers in the set $\{\lambda+\lambda_k\}$ have with n a greatest common divisor Δ.

THEOREM 1. *If Δ is a divisor of the simple number n, and if λ is any fixed number prime to n, and if $\lambda_k (k=1, 2, \cdots, \phi(n))$ runs over a complete set of numbers $<n$ and prime to n, then there are $\phi_2(n/\Delta)$ multiples of Δ prime to n/Δ in the set $\{\lambda+\lambda_k\}$.*

PROOF. Let $n=\Delta\prod_{i=1}^{h} p_i$. Instead of the set $\{\lambda_k\}$ consider the set

$$1, 2, 3, \cdots, n. \tag{I}$$

The set

$$\lambda+1, \lambda+2, \lambda+3, \cdots, \lambda+n, \tag{II}$$

taken modulo n, is the set (I) in some order. In the set (II) there are n/Δ multiples of Δ, $\phi(n/\Delta)$ of which are prime to n/Δ. But this result applies to the set (II) instead of the desired set $\{\lambda+\lambda_k\}$. We must therefore exclude those multiples of Δ from the set (II) which have arisen from the addition of λ to those numbers of the set (I) which do not belong to the set $\{\lambda_k\}$. It is clear that none of the $\phi(n/\Delta)$ multiples of Δ mentioned above were obtained by adding λ to numbers not prime to Δ. We have then to exclude only multiples of Δ obtained by adding λ to numbers not prime to $\prod_{i=1}^{h} p_i$.

In (I) there are n/p_r multiples of p_r. Adding λ to each of these, we obtain a subset (II_r) of (II),

* Dickson, *History of the Theory of Numbers*, Chap. 19.

$$(\mathrm{II}_\nu) \qquad p_\nu + \lambda,\ 2p_\nu + \lambda,\ 3p_\nu + \lambda,\ \cdots,\ \frac{n}{p_\nu}p_\nu + \lambda.$$

Taken modulo n/p_ν, the set (II_ν) is a complete set of incongruent residues. Hence in this set there are $(n/p_\nu)\Delta$ multiples of Δ, not all of which, however, are prime to n/Δ. In fact in the set (II_ν) there are $n/(p_\nu\Delta p_1)$ multiples of p_1, $n/(p_\nu\Delta p_2)$ multiples of p_2 etc. to be excluded. But this excludes twice the multiples of p_1p_2, p_1p_3, $\cdots$, etc. These have to be restored once. By the well known principle of cross-classification, we find that the number of multiples of Δ prime to n/Δ in the set (II_ν) is

$$\frac{n}{p_\nu\Delta} - \sum\frac{n}{p_\nu\Delta p_i} + \sum\frac{n}{p_\nu\Delta p_ip_j} - \cdots = \phi\left(\frac{n}{p_\nu\Delta}\right).$$

As ν runs from 1 to h, we get subsets (II_1), (II_2), $\cdots$, (II_h) of the set (II). The total amount we must subtract, at this stage, from $\phi(n/\Delta)$ to allow for numbers in the set (I) not prime to n is

$$\sum_{\nu=1}^{h}\phi\left(\frac{n}{p_\nu\Delta}\right).$$

But again this excludes twice those multiples of Δ in the set (II) which correspond to multiples of p_1p_2, p_1p_3, $\cdots$, in the set (I). These multiples must be restored once and, using once more the principle of cross-classification, we find that the number of multiples of Δ prime to n/Δ in the set $\{\lambda+\lambda_k\}$ is given by

$$\phi\left(\frac{n}{\Delta}\right) - \sum\phi\left(\frac{n}{p_i\Delta}\right) + \sum\phi\left(\frac{n}{p_ip_j\Delta}\right) - \cdots$$

$$= \sum_{\delta\mid n/\Delta}\phi\left(\frac{n}{\Delta\delta}\right)\mu(\delta) = \phi_2\left(\frac{n}{\Delta}\right),$$

by (2). Hence the theorem is proved.

The preceding theorem is not true for non-simple numbers. The corresponding theorem for Euler's $\phi(n)$ states that the number of numbers $m<n$ such that $(m, n)=\Delta$, is $\phi(n/\Delta)$. This is true for all n. The simple proof of this theorem cannot be extended to Theorem 1.

It is important to notice that the result of Theorem 1 is independent of the choice of λ. As λ runs over all the numbers

in the set $\{\lambda_k\}$ one obtains a matrix $\|a_{ij}\| = \|\lambda_i + \lambda_j\|$ in which by Theorem 1 there are $\phi(n)\,\phi_2(n/\Delta)$ multiples of Δ prime to n/Δ.

The above result furnishes a ready method of obtaining the explicit formula for the discriminant* of the cyclotomic equation $Q_n(x) = 0$, whose roots are the primitive nth roots of unity without repetition, for n a simple number. The following lemma will enable us to obtain the discriminant for a general n.

LEMMA. *If $f(x) = g(x^m)$ is a polynomial of degree k in x^m, and if the discriminant of $g(x)$ is D_g, then the discriminant of $f(x)$ is*

$$D_f = a_k^{2} \cdot m^{mk} D_g^{m},$$

where a_k is the constant term of $g(x)$.

PROOF. Let $\rho_i (i = 1, 2, \cdots, k)$ be the roots of $g(x) = 0$, and let one of the values of $\rho_i^{1/m}$ be θ_i. Then all the roots of $f(x) = 0$ are given by $\theta_i \epsilon^r$, where ϵ is a primitive mth root of unity and $r = 1, 2, \cdots, m$. Then the discriminant of $f(x)$ can be written

$$D_f = \prod_{i<j\leqq k} \prod_{r,s=1}^{m} (\theta_i \epsilon^r - \theta_j \epsilon^s)^2 \cdot \prod_{i=1}^{k} \prod_{r<s\leqq m} (\theta_i \epsilon^r - \theta_i \epsilon^s)^2.$$

The first product is

$$\prod_{i<j\leqq k} \prod_{r=1}^{m} [(\theta_i \epsilon^r)^m - \theta_j^m]^2 = \prod_{i<j\leqq k} (\theta_i^m - \theta_j^m)^{2m} = \prod_{i<j\leqq k} (\rho_i - \rho_j)^{2m} = D_g^m.$$

For the second product we have

$$\prod_{i=1}^{k} \theta_i^{2m} \prod_{r<s\leqq m} (\epsilon^r - \epsilon^s)^2 = \prod_{i=1}^{k} \rho_i^2 \prod_{r<s\leqq m} (\epsilon^r - \epsilon^s)^2 = a_k^2 \prod_{r<s\leqq m} (\epsilon^r - \epsilon^s)^{2k}.$$

The product

$$\prod_{r<s\leqq m} (\epsilon^r - \epsilon^s)^2$$

* Rados, Journal für Mathematik, vol. 131 (1906), pp. 49–55, has calculated the discriminant using the derivative definition. The reduction to a simple number would have simplified his proof considerably.

is the discriminant of x^m-1, which is known to be m^m. Hence the lemma follows at once.

We shall proceed to find the discriminant of Q_n for n a simple number.

THEOREM 2. *If n is simple, the discriminant of $Q_n(x)=0$ is given by*

$$D_n = (-1)^{\phi(n)/2} \prod_{i=1}^{t} p_i^{\phi(n/p_i)\phi_2(p_i)}.$$

PROOF. Since the roots of $Q_n(x)=0$ are $e^{2\pi i\lambda_k/n}$ we have

$$\begin{aligned} D_n &= \prod_{\lambda_k<\lambda_m} (e^{2\pi i\lambda_k/n} - e^{2\pi i\lambda_m/n})^2 \\ &= (-1)^{\phi(n)/2} \prod_{\lambda_k\neq\lambda_m} (e^{2\pi i\lambda_k/n} - e^{2\pi i\lambda_m/n}) \\ &= [\prod e^{2\pi i\lambda_k/n}]^{\phi(n)-1} \prod_{\lambda_k\neq\lambda_m} (1 - e^{2\pi i\lambda_m/n}). \end{aligned}$$

The first product is unity, since $Q_n(0)=1$. To evaluate the second product we observe that there is a one to one correspondence between the numbers $\{\lambda_m-\lambda_k\}$ (mod n) and the elements of the matrix discussed above. The condition $\lambda_m\neq\lambda_k$ excludes from the matrix all the multiples of n. For a divisor Δ of n, the factors of the second product may be grouped into sets of $\phi(n/\Delta)$ elements each, in which $\lambda_m-\lambda_k$ runs over all the numbers prime to n/Δ. By Theorem 1 there are $\phi(n)\phi_2(n/\Delta)/\phi(n/\Delta)=\phi(\Delta)\phi_2(n/\Delta)$ such sets for each $\Delta<n$. Hence

$$D_n = (-1)^{\phi(n)/2} \prod_{\Delta\neq n} [Q_{n/\Delta}(1)]^{\phi(\Delta)\phi_2(n/\Delta)}.$$

It is known* that $Q_n(1)$ is p or 1 according as m is a power of a prime p, or not. Accordingly we have

$$D_n = (-1)^{\phi(n)/2} \prod_{i=1}^{t} p_i^{\phi(n/p_i)\phi_2(p_i)},$$

which is the theorem.

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* N. Trudi, Atti Accademia Napoli, vol. 3 (1866–8), pp. 20–29; Netto, *Vorlesungen über Algebra*, vol. 1, p. 357.

To get the discriminant for a general n we make use of the relation*

$$Q_n(x) = Q_{n_0}(x^m), \tag{3}$$

where $n = n_0 m$, and n_0 is the largest simple factor of n. Hence applying our lemma with $k = \phi(n_0)$ and $a_k = 1$, we have

$$D_n = D_{n_0}^m \cdot m^{m\phi(n_0)} = (-1)^{m\phi(n_0)/2} m^{m\phi(n_0)} \prod_{i=1}^{t} P_i^{m\phi(n_0/p_i)\phi_2(p_i)}.$$

But, since $\phi(n_0/p_i)\phi_2(p_i) = \phi(n_0) - \phi(n_0)/(p_i - 1)$ and $m\phi(n_0) = \phi(n)$, the discriminant can be written in the following general form:

$$D_n = (-1)^{\phi(n)} \frac{m^{\phi(n)}}{\prod_{i=1}^{t} P_i^{\phi(n)/(p_i-1)}}.$$

We shall next consider the resultant $R_{m,n}$ of any Q_m and Q_n. First let m and n be simple numbers ($m \neq n$). Then we have

THEOREM 3. *If m, n are simple numbers,† and $m < n$, then*

$$R_{m,n} = \begin{cases} p^{\phi(m)}, & \text{if } n/m = p, \\ 1, & \text{if } n/m \neq p, \end{cases}$$

where p is a prime.

PROOF. Let $(m, n) = d$, and $m = m_1 d$, $n = n_1 d$. Then if λ and λ' run over numbers prime to m and n, respectively, we have

$$R_{m,n} = \prod (e^{2\pi i\lambda/m} - e^{2\pi i\lambda'/n}) = \left[\prod e^{2\pi i\lambda/n}\right]^{\phi(n)} \cdot \prod (1 - e^{2\pi i(\lambda' m - \lambda n)/(mn)}).$$

The product $\left[\prod e^{2\pi i\lambda/m}\right]^{\phi(n)}$ is equal to 1. The fractions occurring in the last product may be written $(\lambda' m_1 - \lambda n_1)/(m_1 n_1 d)$. The numerators are all prime to $m_1 n_1$ but may not be prime to d. Modulo d, the numbers $\lambda' m$ and λn run, respectively, $\phi(m_1)$ and $\phi(n_1)$ times over the complete set of numbers prime to d. If δ is any divisor of d, then by Theorem 1, there are

* Trudi, Annali di Matematica, (2), vol. 2 (1868–9), pp. 160–2; Netto, loc. cit.

† $R_{1,2} = -R_{2,1} = 2$. For all other values of m, n, $R_{m,n} = R_{n,m}$.

$\phi(m_1)\phi(n_1)\phi(d)\phi_2(d/\delta)$ multiples of δ prime to d/δ in the set $\lambda' m_1 - \lambda n_1$. Hence

$$R_{m,n} = \prod_{\delta \mid d} [Q_{m_1 n_1 d/\delta}(1)]^{\phi(\delta)\phi_2(d/\delta)}.$$

But, by Trudi's theorem quoted above concerning $Q(1)$, the factors of the above product are equal to unity for $\delta < d$. For dm_1n_1/δ cannot be a prime. Hence

$$R_{m,n} = Q_{m_1 n_1}^{\phi(d)}(1).$$

Again, by Trudi's theorem, this expression is unity except when m_1n_1 is a prime p, that is, when $n/m = p$. In this case $R_{m,n} = p^{\phi(m)}$. Hence the theorem.

To find the resultant $R_{m,n}$, where m and n are any distinct integers, we may proceed as follows. Let $m = m_0\mu$ and $n = n_0\nu$ where m_0 and n_0 are the largest simple divisors of m and n. Let also $(\mu, \nu) = d$, $(m_0, \nu/d) = d_1$ and $(n_0, \mu/d) = d_2$. Finally let r_k and s_l be the roots of $Q_{m_0}(x) = 0$ and $Q_{n_0}(x) = 0$; let ρ_k be one of the values of $r_k^{1/\mu}$ and σ_l be one of the values of $s_l^{1/\nu}$. Then, by (3), all the roots of $Q_m(x) = 0$ are given by $\rho_k \epsilon_\mu^i$, where ϵ_μ is a primitive μth root of unity, $k = 1, 2, \cdots, \phi(m_0)$, and $i = 1, 2, \cdots, \mu$. Similarly the roots of $Q_n(x) = 0$ are given by $\sigma_l \epsilon_\nu^j$, where ϵ_ν is a primitive νth root of unity, $l = 1, 2, \cdots, \phi(n_0)$, and $j = 1, 2, \cdots, \nu$. Hence we may write

$$\begin{aligned} R_{m,n} &= \prod_{k=1}^{\phi(m_0)} \prod_{l=1}^{\phi(n_0)} \prod_{i=1}^{\mu} \prod_{j=1}^{\nu} (\rho_k \epsilon_\mu^i - \sigma_l \epsilon_\nu^j) \\ &= \prod_{k,l} \prod_{i=1}^{\mu} (\rho_k \epsilon_\mu^{\nu i} - \sigma_l^\nu) = \prod_{k,l} \prod_{i=1}^{\mu/d} (\rho_k \epsilon_\mu^{\nu i d} - s_l)^d \\ &= \prod_{k,l} (r^{\nu/d} - s^{\mu/d})^d = \prod_{k=1}^{\phi(m_0/d_1)} \prod_{l=1}^{\phi(n_0/d_2)} (\bar{r}_k - \bar{s}_l)^{d\phi(d_1)\phi(d_2)}, \end{aligned}$$

where $\bar{r}$ and $\bar{s}$ are respectively the roots of $Q_{m_0/d_1}(x) = 0$ and $Q_{n_0/d_2}(x) = 0$. Hence

$$R_{m,n} = R_{m_0/d_1,\ n_0/d_2}^{d\phi(d_1)\phi(d_2)}.$$

Since m_0/d_1 and n_0/d_2 are simple numbers we may apply Theorem 3 and obtain the following result.

THEOREM 4. *Let $m=m_0\mu$ and $n=n_0\nu$ be any positive integers, where m_0 and n_0 are the largest simple divisors of m and n. Let $(\mu, \nu)=d$, $(m_0, \nu/d)=d_1$, $(n_0, \mu/d)=d_2$, and* $m_0d_2<n_0d_1$. Then*

$$R_{m,n}=\begin{cases} p^{d\phi(m_0)\phi(d_2)}, & \text{if } n_0d_1/(m_0d_2)=p, \\ 1, & \text{if } n_0d_1/(m_0d_2)\neq p. \end{cases}$$

Since

$$\prod_{\delta/n} Q_\delta(x) = x^n - 1,$$

and since the discriminant of a product of several polynomials is equal to the product of their discriminants times the product of the squares of the resultants of the polynomials taken two at a time, we have the following identity for n a simple number:

$$n^n = \prod_{\delta/n}\left[\left(\frac{n}{\delta}\right)^{2\phi(\delta)} \prod_{p_i/\delta} p_i^{\phi(\delta/p_i)\phi_2(p_i)}\right].$$

If n is not a simple number, the conditions of Theorem 4 are so complicated that the corresponding identity cannot be easily expressed. The following table which gives the discriminants and the resultants belonging to the divisors of 72 illustrates this identity, and gives some idea of the magnitudes under discussion.

δ	1	2	3	4	6	8	9	12	18	24	36	72	D_δ
1		2	3	2	1	2	3	1	1	1	1	1	1
2			1	2	3	2	1	1	3	1	1	1	1
3				1	2^2	1	3^2	2^2	1	2^2	1	1	3
4					1	2^2	1	3^2	1	1	3^2	1	2^2
6						1	1	2^2	3^2	2^2	1	1	3
8							1	1	1	3^4	1	3^4	2^8
9								1	2^6	1	2^6	2^6	3^9
12									1	2^4	3^4	1	$2^4 3^2$
18										1	2^6	2^6	3^9
24											1	3^8	$2^{16} 3^4$
36												2^{12}	$2^{12} 3^{18}$
72													$2^{48} 3$

$R_{m,n}$ is found in the row marked m and the column marked n. The product of the discriminants times the squares of the resultants listed above is found to be $2^{216}\,3^{144}=72^{72}$.

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* This inequality is imposed merely for definiteness and may be reversed at will by interchanging m and n. See the previous footnote.

PROBABILITY AS EXPRESSED BY ASYMPTOTIC LIMITS OF PENCILS OF SEQUENCES*

BY E. L. DODD

1. *Introduction.* From time to time certain conceptions and theorems of mathematics need a restatement. Especially true is this in a subject like probability throughout which is woven so much non-mathematical material. This paper, then, is an attempt to depict probabilities, initially discrete, by the use of aggregates of sequences, in language strictly mathematical, but as simple as possible. The treatment will resemble not so much that of von Mises,† who uses single sequences, as that of Borel,‡ Lomnicki,§ and Steinhaus.‖ But it will differ from the latter two in avoiding the concept of measure, and indeed from all in the approach to the subject. The asymptotic limit introduced in this paper involves a notion resembling that involved in the phrase *almost everywhere.*

In pure mathematics, the word *probability* may be taken¶ to signify simply the ratio of the number of objects in a subset to the number in the set, so long as discrete or arithmetic probability is being considered. It is, indeed, as far outside the field of mathematics to determine whether two events are equally likely as to determine whether two bodies have the same mass. Even in the applications, the role of pure mathematics is merely to count expeditiously the elements of sets and subsets, or more generally to determine certain measures of sets, which are believed by competent judges to depict adequately situations in the external world. It is generally believed, for example,

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* Presented to the Society, August 27, 1929.

† *Grundlagen der Wahrscheinlichkeitsrechnung,* Mathematische Zeitschrift, vol. 5 (1919), pp. 52–99; see especially pp. 55–57.

‡ *Les probabilités dénombrables et leurs applications arithmétiques,* Rendiconti di Palermo, vol. 27 (1909), pp. 247–271.

§ *Nouveaux fondements du calcul des probabilités,* Fundamenta Mathematicae, vol. 4 (1923), pp. 34–71.

‖ *Les probabilités dénombrables et leur rapport à la théorie de la mesure,* Fundamenta Mathematicae, vol. 4 (1923), pp. 286–310.

¶ See Lomnicki, loc. cit., p. 37.

that two tosses of an ordinary coin can be described adequately for most purposes by writing *HH*, *HT*, *TH*, *TT*. Granting this, the verdict of mathematics is that 2/4 is the probability of one head and one tail. On the other hand, if a certain misshapen coin is thought to *favor heads two to one*, the following nine two-element sequences can be written: *HH*, *HH*, *HT*; *HH*, *HH*, *HT*; *TH*, *TH*, *TT*, and the probability of one head and one tail on this hypothesis is 4/9. A supplementary exhibit would commonly appear in columnar form, the first column with two *H*'s and one *T*, each such element followed by a brace enclosing likewise two *H*'s and one *T*. The nature of the problem is unchanged if the coin is declared to favor heads 501 to 499, though the writing of the sequences becomes more laborious.

In trying to *symbolize* the *ordinary coin*, we note that with two different elements, say *H* and *T*, there can be formed exactly 2^s *different* sequences with s elements each. Let m_i be the number of *H*'s in the ith sequence, $i = 1, 2, \cdots, 2^s$. Then the percentage of the sequences in which m_i/s is close to 1/2, when s is large, is almost 100%. This, which contains the essence of Bernoulli's theorem, is a simple *mathematical fact*, independent of any meanings or implications which philosophers or scientists may attach to the word *probability*. Furthermore, it is a correct mathematical *picture* for the proposition (whether this proposition be true or false) that if you keep on tossing an *ordinary* coin, you are *almost sure* to get heads *just about* half the time. This *pictorial correctness* and the *logical rigor* of conclusions is all that mathematicians can be held responsible for; mathematicians know nothing about the truth or falsity of propositions about coins, the implements of chance, or natural statistical phenomena.

In the so-called* Bernoulli case, the elementary probability remains constant from trial to trial; in the *Poisson case*, it changes; for example, if balls are to be drawn successively from a set of urns with different proportions of white and black balls. In the Poisson case, it becomes eventually *almost certain* that the ratio of occurrences (white balls) to trials will closely approximate the *average* of the given probabilities for the occurrence (white ball) of the event on a single trial. Make the

* Rietz, *Mathematical Statistics*, pp. 27–35, in particular, p. 28.

given probabilities all identical, and the Bernoulli case is obtained.

2. *Pencils of Sequences.* Let there be n objects, not necessarily numbers, $a_1, a_2, \cdots, a_n$, which we may think of as arranged in a column rather widely spaced, with a brace, $\{$, following each object. Associated with a_i, and placed in a column to the right of the brace, let there be n_i objects $b_{i1}, b_{i2}, \cdots, b_{in_i}$, for $i=1, 2, \cdots, n$. Such a set of objects we call a *brace.* Also $a_1, a_2, \cdots, a_n$ form the *first brace.* In the same way let each b_{ij} be associated with n_{ij} objects designated by c_{ijk} and so on. From these, sequences may be formed, of which

$$a_i, b_{ij}, c_{ijk}, \cdots$$

is typical. The aggregate of such sequences we call a *pencil of sequences,* and each object such as a_i, b_{ij}, $c_{ijk}, \cdots$ an *element.* An exhibit of decimal fractions* or dyadic fractions would form a pencil.

Frequently it is convenient to take the braces in a given column all alike. That is n_{ij} and b_{ij} do not change with i. In this case a brace characterizing a column will be called a *cross section of the pencil.* And, in contradistinction, the sequences will be called the *longitudinal sections.*

From a given pencil of sequences of objects, corresponding pencils of sequences of numbers can be constructed. Thus, if a sequence is A, A', A', $A, \cdots$, with A perhaps signifying *success,* and A' *failure,*—we can write as a number sequence: 1, 1, 1, 2, $\cdots$, to count the A's; or 1, 1/2, 1/3, 2/4, $\cdots$, to get the relative frequency of the A's. In general, probability problems lead to such numerical sequences. The sth term of such a sequence is a function of s.

3. *Asymptotic Limits.* DEFINITION. For a pencil, or a finite number of pencils of sequences, and a given number s of terms in a sequence, let there be $N(s)$ functions, one for each sequence, $f_1(s), f_2(s), \cdots$, which collectively will be designated $f(s)$. Given $\epsilon>0$, small at pleasure, suppose that $m(s, \epsilon)$ of these functions satisfy

$$|f_i(s) - L| < \epsilon, \tag{1}$$

* See Borel, loc. cit., p. 258.

where L is a constant with respect to s and t, and where

$$\lim_{s\to\infty} \frac{m(s,\epsilon)}{N(s)} = 1. \tag{2}$$

We call L an asymptotic limit or an almost limit, and write

$$\operatorname*{alim}_{s\to\infty} f(s) = L. \tag{3}$$

This means that with $\epsilon>0$, $\eta>0$, small at pleasure, it is possible to find s' so that if $s>s'$, we have $0\leqq 1-m/N<\eta$.

A number of writers have recognized that problems of probability often lead to a relation resembling a limit; for example, when it becomes almost certain or asymptotically certain that an empirical ratio will approximate a theoretic ratio or *probability*. Cantelli* capitalizes the l of lim, writing Lim for his near-limits.

PROPERTIES. Analogous to properties of ordinary limits are properties of these asymptotic limits. For example, if with a pencil of sequences we define two functions $f(s)$ and $g(s)$ having asymptotic limits L and L', then

$$\operatorname*{alim}_{s\to\infty} [f(s) + g(s)] = L + L'. \tag{4}$$

For if $m'(s, \epsilon)$ is the number of sequences for which $|f_t(s)-L|>\epsilon/2$, and $m''(s, \epsilon)$ is the number for which $|g_t(s)-L'|>\epsilon/2$, then the number for which $|f_t(s)+g_t(s)-(L+L')|>\epsilon$ cannot exceed $m'(s, \epsilon)+m''(s, \epsilon)$. But the ratio of this sum to $N(s)$ approaches zero as a limit. In a similar manner, we prove the theorem: If $F(x, \cdots, y)$ is continuous at

$$x = L, \cdots, y = L';$$

and alim $f(s)=L, \cdots,$ alim $g(s)=L'$, it follows that†

$$\operatorname*{alim}_{s\to\infty} F[f(s), \cdots, g(s)] = F[L, \cdots, L']. \tag{5}$$

4. *The Bernoulli and Poisson Theorems.* Of the theorems of probability few are more important than the Bernoulli theorem

* *La tendenza ad un limite nel senso del calcolo della probabilità*, Rendiconti di Palermo, vol. 4 (1916), pp. 191–201.

† See Cantelli, loc. cit., p. 199.

with its extensions and its inverse. For the Bernoulli theorem all the *cross sections* of the pencil are *identical*; for the Poisson theorem, the cross sections are in general different. The Poisson theorem, in the language here used, is then a statement that the *average cross section* of a pencil gives a correct estimate of nearly all the *longitudinal sections*; that is, with the exception of a percentage of the latter, which evanesces with increasing s. To express this more fully, suppose that to form a pencil, n of the A's, B's, $\cdots$, K's are taken for the first column, n' of these elements for each identical brace in the second column, n'' for each identical brace in the third column, and so on. Let p_1 be the ratio of the number of A's in the first column to n, let p_2 be the ratio of the number of A's in each brace of the second column to n', and so on. Let $f_t(s)$ be the number of A's among the first s elements of the tth sequence. The Poisson theorem is then the statement that if

$$\lim_{s\to\infty} \frac{1}{s}[p_1 + p_2 + \cdots + p_s] = p, \tag{6}$$

it follows that

$$\operatorname*{alim}_{s\to\infty} \frac{f(s)}{s} = p. \tag{7}$$

Not only is this true for the A's, but the analogous relation holds for the B's, $\cdots$, K's, *simultaneously*, by the addition theorem for asymptotic limits. Hence, asymptotically, that is, excepting an evanescent percentage of the sequences, the distribution of A's, B's, $\cdots$, K's in the sequences conforms to the ratios set up by the *average cross section* of the pencil. The result is unchanged if a finite number of the cross sections are altered or are *unknown*, as this would not effect the value of p. At this point there is an analogy between this *direct* form of probability, and the *indirect* or *inverse* based upon *unknown a priori* probabilities, which we consider next.

5. *The Inverse to Bernoulli's Theorem.* An urn contains b balls of which $a = pb$ are white, where a and p are unknown. In s drawings of a ball with replacement, r white balls have appeared. Then r/s may be accepted as an estimate of the unknown probability p, of getting a white ball in one drawing.

Indeed, with $\eta>0$, small at pleasure, there is an *a posteriori* probability greater than $1-\eta$, when s is sufficiently large, that $|r/s-p|<1/b$, at least, if it can be assumed that there was a positive *a priori* probability, not necessarily known, for each of the $(b+1)$ possible ways of filling the urn. By means of some such schema, some precision can be given to the Bernoulli inverse. But even here are mentioned various probabilities capable of exciting philosophic perplexities. So it seems desirable to express the mathematical relations involved without using the word *probability*.

The inverse theorem, in discrete form, makes use of certain unknown *a priori* probabilities, which in the formulation which follows will be represented by proportional arbitrary* constants, N_a. The *unknown cause* is the value a/b of a probability. After *r successes in s trials*, the value r/s is attributed to a/b. The ϵ used is the *margin* between the discrete values considered for the probability a/b. In exact language, and thus in language unencumbered by any notions of probability, the theorem may be stated as follows.

Let ϵ and a whole number b be so chosen that

$$\frac{1}{b} \leqq \epsilon, \tag{8}$$

and let $(b+1)$ whole numbers, $N_0, N_1, \cdots, N_b$, be arbitrarily taken. For each a in turn, $a=0, 1, 2, \cdots, b$, let N_a pencils of sequences be constructed, each sequence with s elements, by putting into each brace a of the A's, and $b-a$ of the not-A's. Let $n(r, s)$ be the number of these sequences with exactly r of the A's among its s elements and let $m(r, s, \epsilon)$ be the number of such sequences for which

$$\left| \frac{r}{s} - \frac{a}{b} \right| \leqq \epsilon. \tag{9}$$

Then

$$\lim_{s\to\infty} \frac{m(r,s,\epsilon)}{n(r,s)} = 1, \tag{10}$$

* See von Mises, *Fundamentalsätze der Wahrscheinlichkeitsrechnung*, Mathematische Zeitschrift, vol. 4 (1919), pp. 1–97, in particular, p. 83.

uniformly for r. In other words, given $\eta>0$, small at pleasure, it is possible to find s' so that if $s>s'$, and r *thereafter* is given any of the values $0, 1, \cdots, s$, we have

$$\left|\frac{m(r,s,\epsilon)}{n(r,s)}-1\right|<\eta. \tag{11}$$

This, indeed, permits r/s to have values near zero or unity where the approximation of the Laplacean integral is not so good. On account of (8), at least one integral value a satisfies (9).

If, moreover, it is understood that b is to be chosen after ϵ, the conclusion may be stated as follows: For the tth sequence which contains exactly r of the A's, let $f_t(r, s)$ be the value of a which characterizes and initiates the sequence, and let a_{rs} be a collective symbol for $f_t(r, s)$. Then

$$\operatorname*{alim}_{s\to\infty}\left(\frac{a_{rs}}{b}-\frac{r}{s}\right)=0, \tag{12}$$

uniformly for r.

6. *Conclusion.* The foregoing is perhaps sufficient to show that, without entering into the somewhat intricate theory of measure, we may express accurately some of the most important theorems of probability in just about the usual language of mathematics, at least avoiding such notions as *cases equally likely*, a concept which seems to have significance for the external world rather than for mathematics. Of course, after the mathematical framework is understood, there is no more harm in employing the usual language of probability than in speaking of a sphere in n dimensions.

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SOME PROPERTIES OF SPHERICAL HARMONICS*

BY H. BATEMAN

A Newtonian potential $V(x, y, z)$ can often be derived from a four-dimensional potential $W(x, y, z, w)$ by forming the definite integral

$$V = \frac{1}{\pi}\int_{-\infty}^{\infty} W dw.$$

Thus, if W is the reciprocal of $(x-x_0)^2+(y-y_0)^2+(z-z_0)^2+(w-w_0)^2$, where x_0, y_0, z_0, w_0 are real constants, the Newtonian potential V is the inverse of the distance between the points (x, y, z) and (x_0, y_0, z_0). There is thus a simple correspondence between the *charges* giving rise to the two potentials, the point charge in the three-dimensional space, S_3, being simply the projection of the corresponding point charge in the four-dimensional space S_4. With suitable restrictions this method of projection may be applied to surface distributions of charge in S_4 and we shall consider in particular the case of a continuous distribution over the spherical surface

$$x^2 + y^2 + w^2 = a^2, \quad z = 0,$$

when the surface density depends only on x^2+y^2. In this case

$$W = \int_0^{\pi}\int_0^{2\pi} (a^2/R^2) f(\cos\theta)\sin\theta d\theta d\phi,$$

where $f(\cos\theta)$ is the function giving the law of density and

$$R^2 = (w - a\cos\theta)^2 + z^2 + (x - a\sin\theta\cos\phi)^2 + (y - a\sin\theta\sin\phi)^2.$$

To reduce the integral to a simpler form we write

$$\cos\theta = \zeta, \qquad x^2 + y^2 = \rho^2, \qquad \rho^2 + w^2 = \tau^2;$$

then

$$W = (\pi a/\tau) U(X, Y, Z),$$

where

* Presented to the Society, June 20, 1929.

$$U = \int_{-1}^{1} [(\zeta - Z)^2 + X^2 + Y^2]^{-1/2} f(\zeta) d\zeta ,$$

$$Z = ws^2/(2a\tau^2), \quad X^2 + Y^2 + Z^2 = (s^4 - 4a^2\rho^2)/(4a^2\tau^2),$$

$$s^2 = x^2 + y^2 + z^2 + w^2 + a^2 .$$

Now, when (X, Y, Z) are regarded as rectangular coordinates in a new space S_3*, the function U is the Newtonian potential of a rod of line density $f(\zeta)$ and so it is advantageous to introduce spheroidal coordinates ξ, η, of the prolate type, by means of the equations

$$\cos \xi \cosh \eta = Z = ws^2/(2a\tau^2) ,$$

$$\sin \xi \sinh \eta = (X^2 + Y^2)^{1/2} = \rho(s^4 - 4a^2\tau^2)^{1/2}/(2a\tau^2) .$$

We then find that

$$\cos \xi = w/\tau, \qquad \cosh \eta = s^2/(2a\tau) .$$

If $P_n(\mu)$, $Q_n(\sigma)$ are the usual Legendre functions, the potential

$$U = Q_n(\cosh \eta) P_n(\cos \xi)$$

gives rise to the four-dimensional potential

$$W = \pi(a/\tau) Q_n(s^2/(2a\tau)) P_n(w/\tau) .$$

This is a particular case of the theorem that if $H(x, y, w)$ is a homogeneous function of x, y, w of degree $-n-1$ such that

$$\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} + \frac{\partial^2 H}{\partial w^2} = 0,$$

and T is a function of z and τ satisfying

$$\frac{\partial^2 T}{\partial \tau^2} - \frac{2n}{\tau} \frac{\partial T}{\partial \tau} + \frac{\partial^2 T}{\partial z^2} = 0, \tag{1}$$

the product $W = HT$ is a four-dimensional potential. It should be noted that the equation (1) is satisfied by $\tau^n P_n(s^2/(2a\tau))$ and $\tau^n Q_n(s^2/(2a\tau))$. To ascertain the nature of the function V derived from W we note that when $\rho = 0$ the potential V becomes

$$a\int_{-\infty}^{\infty}\frac{dw}{w}\,Q_n\left(\frac{z^2+a^2+w^2}{2aw}\right)=\frac{a}{2}\int_{-\infty}^{\infty}dw\int_{-1}^{1}\frac{2aP_n(\mu)d\mu}{z^2+a^2+w^2-2a\mu w}$$

$$=a^2\int_{-1}^{1}d\mu\int_{-\infty}^{\infty}\frac{P_n(\mu)dw}{(w-a\mu)^2+a^2(1-\mu^2)+z^2}$$

$$=a^2\pi\int_{-1}^{1}[z^2+a^2(1-\mu^2)]^{-1/2}P_n(\mu)d\mu$$

$$=2a\pi C_m\cdot q_{2m}(|z|/a), \qquad (n=2m),$$

$$=0, \qquad (n=2m+1),$$

where

$$C_m=\frac{1\cdot 3\cdot\cdots\cdot(2m-1)}{2\cdot 4\cdot\cdots\cdot(2m)},$$

and where

$$q_{2m}(\sigma)=\frac{\pi^{1/2}\Gamma(2m+1)}{2^{2m+1}\Gamma(2m+3/2)}(1+\sigma^2)^{-m-1/2}\cdot F\left(m+\frac{1}{2},\ m+\frac{1}{2};2m+\frac{3}{2};\frac{1}{1+\sigma^2}\right)$$

is the type of Legendre function used for the standard harmonics associated with an oblate spheroid, the notation being that used in Lamb's *Hydrodynamics*, 5th edition, p. 124. The change of the order of integration in the repeated integral in the above analysis is easily justified when $z^2>\epsilon^2>0$ because Weierstrass' test may be used to establish the uniform convergence of the infinite integral in the range $-1\leqq\mu\leqq 1$.

The function V is the potential of a circular disc charged with a surface density $[a^2/(a^2-\rho^2)]^{1/2}P_{2m}[(1-\rho^2/a^2)^{1/2}]$, the charges on the two sides being equal. By introducing the spheroidal coordinates σ, μ, defined by the equations

$$z=a\mu\sigma, \quad \rho=a[(1-\mu^2)(1+\sigma^2)]^{1/2},$$

the potential V may be expanded in a unique manner in a series of spheroidal harmonics of type $A_nP_n(\mu)q_n(\sigma)$; and by putting $\mu=1$ it is easily seen that V is identical with the harmonic $2a\pi C_mP_{2m}(\mu)q_{2m}(\sigma)$. Hence

$$(2)\qquad \int_{-\infty}^{\infty}\frac{dw}{\tau}Q_{2m}(s^2/(2a\tau))P_{2m}(w/\tau)=2\pi C_mP_{2m}(\mu)q_{2m}(\sigma).$$

This is a relation between spheroidal harmonics of the prolate and oblate types.

On the disc we have $\sigma = 0$, and since $2q_{2m}(0) = \pi C_m$, the value of V is

$$V_0 = \pi^2 a C_m{}^2 P_{2m}(\mu).$$

Another relation, which is obtained by the method of projection from S_4, may be mentioned here. If $r^2 = x^2 + y^2 + z^2$, $\rho^2 = x^2 + y^2$,

$$(3)\quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{dw}{(r^2 + w^2)^{n+1}} P_n\left(\frac{z^2 + w^2 - \rho^2}{z^2 + w^2 + \rho^2}\right) = C_n r^{-2n-1} P_{2n}\left(\frac{x}{r}\right).$$

If $w = r \tan \epsilon$, it is easily seen that the integral is of the form $r^{-2n-1}F(z/r)$, where $F(\nu)$ is a polynomial. Since the integral represents a Newtonian potential, $F(\nu)$ must be a constant multiple of $P_{2n}(\nu)$ and the constant multiplier may be identified with C_n by putting $\rho = 0$.

The associated relation

$$(4)\quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{dw}{(r^2 + w^2)^{n+1}} Q_n\left(\frac{z^2 + w^2 - \rho^2}{z^2 + w^2 + \rho^2}\right) = C_n r^{-2n-1} Q_{2n}\left(\frac{z}{r}\right)$$

is more difficult to prove. The integral may be shown by differentiation to be a Newtonian potential and the substitution $w = r \tan \epsilon$ indicates that it is of the form $r^{-2n-1}[A_n Q_{2n}(z/r) + B_n P_{2n}(z/r)]$, where A_n and B_n are constants to be determined. Now $Q_{2n}(0) = 0$, $P_{2n}(0) = (-1)^n C_n$, hence $B_n = 0$ if

$$(5)\quad \int_{-\infty}^{\infty} \frac{dw}{(\rho^2 + w^2)^{n+1}} Q_n\left(\frac{w^2 - \rho^2}{w^2 + \rho^2}\right) = 0.$$

To establish this relation we make use of the expansion

$$(6)\quad \sum_{n=0}^{\infty} h^n Q_n(\nu) = (1 - 2\nu h + h^2)^{-1/2} \sinh^{-1}\left[(\nu - h)(1 - \nu^2)^{-1/2}\right]$$
$$|h| < 1, \quad (-1 < \nu < 1),$$

which is readily derived from the formula* defining $Q_n(\nu)$, namely,

$$(7)\quad \frac{1}{r^{n+1}} Q_n\left(\frac{z}{r}\right) = \frac{(-1)^n}{2^n \cdot n!} \frac{\partial^n}{\partial z^n}\left(\frac{1}{2r} \log \frac{r + z}{r - z}\right).$$

* E. W. Hobson, Proceedings of the London Mathematical Society, (1), vol. 22 (1891), p. 438. (This reference was given to me by G. N. Watson.)

The expansion indicates that $Q_n[(w^2-\rho^2)/(w^2+\rho^2)]$ can be regarded as an odd function of w and so the truth of (5) becomes manifest. To determine the value of A_n we shall first obtain another integral for the same Newtonian potential by making use of the fact that if the function $W=F(x, y, z^2-t^2)$ is a solution of the wave-equation

$$\frac{\partial^2 W}{\partial x^2}+\frac{\partial^2 W}{\partial y^2}+\frac{\partial^2 W}{\partial z^2}=\frac{\partial^2 W}{\partial t^2},$$

the integral

$$V=\frac{1}{2}\int_{t-z}^{t+z} F[x, y, z^2-(t-\tau)^2]d\tau$$

is in many cases a solution of Laplace's equation. Instead of enumerating a set of sufficient conditions to be satisfied by F we shall simply remark that the conditions (indicated by the differentiations under the integral sign) are evidently satisfied in the present case (when n is a positive integer) because the integrand in the integral now to be considered is a polynomial:

$$(8)\qquad \frac{1}{2}\int_{t-z}^{t+z} d\tau[r^2-(t-\tau)^2]^n P_n\left[\frac{\rho^2-z^2+(t-\tau)^2}{\rho^2+z^2-(t-\tau)^2}\right] = (-1)^n C_n r^{-2n-1} Q_{2n}(z/r)\,.$$

The integral evidently represents a function of type $r^{-2n-1}F(z/r)$ and is zero when $z=0$, consequently it represents a constant multiple of $r^{-2n-1}Q_{2n}(z/r)$. The constant multiplier may be determined by differentiating with respect to z and making use of the relation

$$Q'_{2n}(0) = (-1)^n/C_n.$$

This may be proved by differentiating the relation (5) with respect to z and then putting $z=0$ and using the expansion

$$(9)\quad h(1+h^2)^{-3/2}\sinh^{-1}h - 1/(1+h^2) = \sum_{n=0}^{\infty}(-1)^{n+1}h^n/C_n.$$

Multiplying the relation (4) by a^{2n} and summing from $n=0$ to $n=\infty$, we obtain the relation

$$(10)\qquad \int_0^{\phi}\frac{d\theta}{(r_1^2-r_2^2\sin^2\theta)^{1/2}} = \sum_{n=0}^{\infty}(-1)^n C_n(a^{2n}/r^{2n+1})Q_{2n}(z/r)\,,$$

where $r_1^2=(\rho+a)^2+z^2$, $r_2^2=(\rho-a)^2+z^2$, and $r_2\sin\phi=z$. This relation holds for $\rho\geqq a$.

Calling the potential function on the left of (10) v, we note that if (4) is correct, there must be a relation of type

$$v=\frac{1}{\pi}\int_{-\infty}^{\infty}dw\left[a^4+2a^2(z^2+w^2-\rho^2)\right.$$
$$\left.+(r^2+w^2)^2\right]^{-1/2}\sinh^{-1}\left[\frac{a^2+z^2+w^2-\rho^2}{2\rho(z^2+w^2)^{1/2}}\right],\quad \rho\geqq a.$$

To test this relation we first put $\rho=a$. It then becomes

$$\int_0^{\pi/2}\frac{d\theta}{(4a^2+z^2\cos^2\theta)^{1/2}}$$
$$=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{dw}{[(z^2+w^2)(z^2+w^2+4a^2)]^{1/2}}\sinh^{-1}\left[\frac{(z^2+w^2)^{1/2}}{2a}\right].$$

This relation may be checked by expansion in powers of $1/a$, making use of the expansion

$$(11)\quad (z^2+w^2+4a^2)^{-1/2}\sinh^{-1}\left[\frac{(z^2+w^2)^{1/2}}{2a}\right]$$
$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)C_n}\frac{(z^2+w^2)^{n+1/2}}{(2a)^{2n+2}}.$$

This will serve as an outline of a method by which the relation (4) may be established.

It should be noticed that by making use of (7) the series on the right of (10) may be expressed in the form

$$(12)\qquad v=J_0\left[a\frac{\partial}{\partial z}\right]\left\{\frac{1}{2r}\log\frac{r+z}{r-z}\right\},$$

and, since*

$$(13)\qquad \frac{1}{2r}\log\frac{r+z}{r-z}=-\frac{\pi}{2}\int_0^{\infty}e^{-kz}Y_0(k\rho)dk,$$

the last expression may be replaced by a definite integral. The appropriate formula is

* Watson, *Bessel Functions*, p. 387.

$$(14)\qquad \begin{cases} v = -\frac{1}{2}\pi \int_0^\infty e^{-kz} J_0(ka) Y_0(k\rho) dk, & (\rho > a > 0), \\ v = -\frac{1}{2}\pi \int_0^\infty e^{-kz} Y_0(ka) J_0(k\rho) dk, & (a > \rho > 0). \end{cases}$$

G. N. Watson has kindly mentioned that the function v is also expressible in the forms

$$(15)\qquad \begin{aligned} v &= -\frac{1}{2}\int_0^\infty dk \int_0^\pi e^{-kz} Y_0(kR) d\phi \\ &= -\frac{1}{2}\int_0^\pi d\phi \int_0^\infty e^{-kz} Y_0(kR) dk, \quad (R^2 = \rho^2 + a^2 - 2\rho a \cos\phi), \end{aligned}$$

the repeated integral being absolutely convergent.

Formula (14) is easily checked by noticing that v is a symmetric function of ρ and a and that a correct result is obtained by putting $\rho = 0$. That the result is also correct for $\rho = a$ is seen by making use of the well known formula*

$$(16)\qquad J_0(ka) Y_0(ka) = -\frac{2}{\pi}\int_0^\infty J_0(2ka \cosh u) du.$$

The equation then becomes

$$\int_0^\infty (4a^2 + \cos^2\theta)^{-1/2} d\theta = \int_0^\infty (z^2 + 4a^2 \cosh^2 u)^{-1/2} du.$$

If we first put $\rho = a$ in v and then make $z = 0$ the result is $\pi/(4a)$ but if we first put $z = 0$ and then make $\rho \to a$ the result is zero, a result which is in agreement with the fact that v is an odd function of z.

It is interesting to note that

$$(17)\qquad \begin{aligned} \int_0^\infty J_0(\lambda a) Y_0(\lambda\rho) d\lambda &= 0, && (\rho > a > 0), \\ &= -1/(2a), && (\rho = a > 0). \end{aligned}$$

The value of the integral for $\rho < a$ has been obtained for me by G. N. Watson. When $0 < \rho < a$ it is

* N. Nielsen, *Handbuch der Theorie der Cylinder Funktionen*, p. 215.

$$-\frac{2}{\pi a}K'\left(\frac{\rho}{a}\right)$$

where K and iK' are the quarter periods of elliptic functions of modulus k. It should be observed that

$$\lim_{\rho \to a-0} \int_0^\infty J_0(\lambda a)Y_0(\lambda\rho)d\lambda = -1/a. \tag{18}$$

When $\rho<0$, $a>0$, $\rho=\sigma e^{i\pi}$ $(\sigma>0)$, $Y_0(\lambda\rho)=Y_0(\lambda\sigma)+2iJ_0(\lambda\sigma)$, the value is

$$-\frac{2}{\pi a}K'\left(\frac{\sigma}{a}\right)+\frac{4i}{\pi a}K\left(\frac{\sigma}{a}\right)$$

for $\sigma<a$ and $[4i/(\pi\sigma)]K(a/\sigma)$ for $\sigma>a$. When $\sigma=a$ the integral is divergent.

Watson remarks that a combination of (15) with (13) gives the simple formula

$$v=\frac{1}{2\pi}\int_0^\pi \frac{d\phi}{r'}\log\frac{r'+z}{r'-z}, \quad (r'^2=z^2+R^2). \tag{19}$$

Two other simple expressions for v may be obtained by using the known formula*

$$Y_0(x) = -\frac{2}{\pi}\int_0^\infty \cos(x\cosh u)du, \qquad (x>0). \tag{20}$$

The result is that

$$v = \text{real part of} \begin{cases} \displaystyle\int_0^\infty du\left[(z+i\rho\cosh u)^2+a^2\right]^{-1/2}, & (\rho \geqq a), \\ \displaystyle\int_0^\infty du\left[(z+ia\cosh u)^2+\rho^2\right]^{-1/2}, & (\rho \leqq a). \end{cases} \tag{21}$$

Here v is the potential corresponding to a surface density proportional to $1/(az)$ on the cylinder $\rho=a$. The discontinuity in the surface density at the plane $z=0$ is responsible for the discontinuity in v. We have actually

* When $\rho>a>0$ the result is easily deduced from Nielsen's formula (12) on p. 193 of his book.

$$(22)\qquad \begin{cases} [\partial v/\partial\rho]_{\rho=a+0} - [\partial v/\partial\rho]_{\rho=a-0} = -1/(az), \\ [v(\rho,z)]_{\rho=a,z=+0} - [v(\rho,z)]_{\rho=a,z=-0} = \pi/(2a). \end{cases}$$

The behavior of the potential would be precisely the same if the surface density proportional to $1/(az)$ were distributed on the sphere $r=a$ instead of the cylinder $\rho=a$. The potential in this case can be expressed in the forms

$$(23)\qquad \begin{cases} v_1 = \sum_{n=0}^{\infty} \dfrac{(-1)^n}{(2n+1)C_n} \dfrac{r^{2n+1}}{a^{2n+2}} P_{2n+1}(z/r), & (r \leqq a), \\ v_1 = \sum_{n=0}^{\infty} \dfrac{(-1)^n}{(2n+1)C_n} \dfrac{a^{2n+1}}{r^{2n+2}} P_{2n+1}(z/r), & (r \geqq a). \end{cases}$$

For $r \leqq a$ we have, indeed, $v_1 = v$. This may be seen by making use of the relation

$$(24)\qquad \frac{1}{2}\int_{t-z}^{t+z} d\tau [r^2 - (t-\tau)^2]^n P_n\left[\frac{\rho^2 - z^2 + (t-\tau)^2}{\rho^2 + z^2 - (t-\tau)^2}\right] = \frac{(-1)^n r^{2n+1}}{(2n+1)C_n} P_{2n+1}\left(\frac{z}{r}\right),$$

which is proved* in the same way as (8) when use is made of the relation

$$(25)\qquad P'_{2n+1}(0) = (-1)^n/[(2n+1)C_n].$$

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* Watson, *Bessel Functions*, p. 180.

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OFFICIAL COMMUNICATIONS

Meetings of the Society have been fixed at the following times and places:

STANFORD UNIVERSITY, CALIFORNIA, April 5, 1930.

Abstracts must be in the hands of Associate Secretary T. M. Putnam, University of California, Berkeley, Calif., not later than March 8.

NEW YORK CITY, April 18-19, 1930.

Abstracts must be in the hands of the Secretary of the Society, Professor R. G. D. Richardson, 501 West 116th St., New York City, not later than March 28. The Committee on Program is arranging a symposium on Differential Geometry, in which Professors Oswald Veblen, L. P. Eisenhart, W. C. Graustein, and others will participate.

CHICAGO, April 18-19, 1930.

Abstracts must be in the hands of Associate Secretary M. H. Ingraham, University of Wisconsin, Madison, Wisconsin, not later than March 28. An invited address will be given by Professor Henry Blumberg on *Methods in point sets and the theory of real functions.*

EUGENE, OREGON, June 20, 1930.

Abstracts must be in the hands of Associate Secretary T. M. Putnam, University of California, Berkeley, Calif., not later than May 23. For publication in advance of the meeting, abstracts must be received on or before May 13.

R. G. D. RICHARDSON, *Secretary of the Society.*

Articles for insertion in the BULLETIN should be addressed to E. R. HEDRICK, Editor of the BULLETIN, University of California at Los Angeles. Reviews should be sent to W. R. LONGLEY, Yale University, New Haven, Conn. Notes should be sent to H. W. KUHN, Ohio State University, Columbus, Ohio.

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Changes of address of members should be communicated at once to the Secretary of the Society, R. G. D. RICHARDSON, 501 West 116th Street, New York.

The initiation fees ($5.00) and the annual dues ($6.00) of members of the Society are payable to the Treasurer of the Society, W. B. FITE, 501 West 116th Street, New York.

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